Assignment 1 Solutions

1) Prove by induction that:

a) **the sum of n rational numbers is rational.**

b) **the product of n rational numbers is rational.**

a)

- Base Case ($n = 2$):
  $\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2} = \frac{p}{q}$, where both p and q are integers. So $\frac{p}{q}$ is rational.

- Inductive Hypothesis:
  Suppose that the given statement is true for $n = k$.
  $\sum_{i=1}^{k} \frac{p_i}{q_i}$ is rational $(\sum_{i=1}^{k} \frac{p_i}{q_i} = \frac{p'}{q'})$.

- Inductive Step:
  We have to prove that the given statement is true for $n = k + 1$.

$$\begin{aligned} \sum_{i=1}^{k+1} \frac{p_i}{q_i} &= \sum_{i=1}^{k} \frac{p_i}{q_i} + \frac{p_{k+1}}{q_{k+1}} \\ &= \frac{p'}{q'} + \frac{p_{k+1}}{q_{k+1}} \quad (\textit{by inductive hypothesis}) \\ &= \frac{p'(q_{k+1}) + q'(p_{k+1})}{q' q_{k+1}} \\ &= \frac{P}{Q} \end{aligned}$$

  where P and Q are integers. So, $\frac{P}{Q}$ is rational.

  This implies that the sum of n rational numbers is rational.

b)

- Base Case ($n = 2$):
  $\frac{p_1}{q_1} \cdot \frac{p_2}{q_2} = \frac{p_1 p_2}{q_1 q_2} = \frac{p}{q}$, where both p and q are integers. So $\frac{p}{q}$ is rational.

- Inductive Hypothesis:
  Suppose that the given statement is true for $n = k$.
  $\prod_{i=1}^{k} \frac{p_i}{q_i}$ is rational $(\prod_{i=1}^{k} \frac{p_i}{q_i} = \frac{p'}{q'})$.

- Inductive Step:
  We have to prove that the given statement is true for $n = k + 1$.

$$\begin{aligned}
\prod_{i=1}^{k+1} \frac{p_i}{q_i} &= \prod_{i=1}^{k} \frac{p_i}{q_i} \cdot \frac{p_{k+1}}{q_{k+1}} \\
&= \frac{p'}{q'} \cdot \frac{p_{k+1}}{q_{k+1}} \quad (\textit{by inductive hypothesis}) \\
&= \frac{p' p_{k+1}}{q' q_{k+1}} \\
&= \frac{P}{Q}
\end{aligned}$$

  where P and Q are integers. So, $\frac{P}{Q}$ is rational.

  This implies that the product of n rational numbers is rational.

2) Prove that n is even if and only if n^3 is even.

- Assume that n is even. We have to prove that n^3 is also even. As n is even, we can write $n = 2k$ for some $k \in \mathbb{Z}$, where $\mathbb{Z}$ represents the set of all integers.

$$\begin{aligned}
n^3 &= n \cdot n \cdot n \\
&= 2k \cdot 2k \cdot 2k \\
&= 8k^3 \\
&= 2(4k^3)
\end{aligned}$$

  As 2 is a factor of n^3, it is obvious that n^3 is even.

- Next, assume that n^3 is even. We have to show that n is even.

  As n^3 is even, it should have 2 as a factor. n^3 is the product of 3 n's and there is no other number in the product. So, the factor 2 of n^3 is contributed by n itself. Thus, it is clear that the number n has a factor 2.

  Since 2 is a factor of n, we can conclude that n is even.

  Thus, n is even if and only if n^3 is even.

3) Prove by induction that $n^3 - n$ is divisible by 3 for all integers $n \geq 1$.

- Base Case ($n = 1$):
  $1^3 - 1 = 0$, which is divisible by 3.

- Inductive Hypothesis:
  Suppose that the given statement is true for $n = k$ (3 divides $k^3 - k$, and so $k^3 - k = 3P$ for some $P \in \mathbb{Z}$).

- Inductive Step:
  Prove that the given statement is true for $n = k + 1$.

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= (k^3 - k) + 3k^2 + 3k \\ &= 3P + 3k^2 + 3k \quad \textit{(by inductive hypothesis)} \\ &= 3(P + k^2 + k), \textit{which is divisible by } 3.\end{aligned}$$

  Thus, $n^3 - n$ is divisible by 3 for all integers $n \geq 1$.

4) Prove by induction that $n^2 - 1$ is divisible by 4 whenever $n > 1$ and n is an odd integer.

REMEMBER that if $a|b$ (*meaning that a divides b*) is true, then (*prop. 1*) $b = aq$, where q is an integer. And if $a|b$ and $a|c$, then (*prop. 2*) $a|b + c$ and (*prop. 3*) $a|b - c$.

Basis case

For $n = 3$,

$$\begin{aligned}&4|n^2 - 1 \\ &4|9 - 1 \\ &4|8 \\ &4|4 \cdot 2\end{aligned}$$

is true with *prop. 1*.

Induction hypothesis

For $n = k$, let assume that $4|k^2 - 1$ is true.

Induction step

For $n = k + 2$,

$$4|(k+2)^2 - 1$$
$$4|k^2 + 4k + 4 - 1$$
$$4|(k^2 - 1) + (4k + 4)$$

With *prop. 2*, we have $4|k^2 - 1$ and $4|4k + 4$. $4|k^2 - 1$ is true because it's the hypothesis and $4|4k + 4 \Rightarrow 4|4(k+1)$ is true base on *prop. 1*.

5) If possible, give an example of a graph with 6 vertices whose degrees are:

a) 2, 3, 3, 4, 4 and 5.

b) 2, 3, 3, 3, 4 and 5.

If it is not possible, explain why.

a) It's not possible since in a graph we must have $\sum_{i=1}^{|V|} deg(v_i) = 2|E|$ and in our case $\sum_{i=1}^{|V|} deg(v_i) = 21$ and it's not possible that $2|E| = 21$

b) The graph of Figure 1 is an example of such a graph.

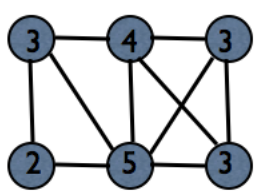

Figure 1: A graph with vertices and degrees 2, 3, 3, 3, 4, 5

6) Let $A = \{a, b\}$ and B be a countable set. Prove or disprove the statement that the set of all *functions* from A to B is countable.

We can represent each function from A to B as an ordered pair: (a, b) where $a \in A$ (the input) and $b \in B$ (the output). All possible pairs can be found by cross product the set A with the set B. Since A and B are countable sets, the cross product of two countable sets results as countable set (proven in class). Q.E.D.

7) Let A be the set of all prime numbers, and B be the set of natural numbers divisible by 5. Prove that the set $C = \{a^b | a \in A$ and $b \in B\} \cup \{b^a | a \in A$ and $b \in B\}$ is countable.

A = the set of all prime numbers = {2, 3, 5, 7, 11, 13, ...}
B = the set of natural numbers divisible by 5 = {5, 10, 15, 20, ...}

$A \subset \mathbb{N}$ and $B \subset \mathbb{N}$, where $\mathbb{N}$ represents the set of all natural numbers.
$\mathbb{N}$ is countable by definition. Since the subset of a countable set is countable, A and B are also countable.

Let $C = D \cup E$ where $D = \{a^b | a \in A$ and $b \in B\}$ and $E = \{b^a | a \in A$ and $b \in B\}$.

Each element in D can be represented as an ordered pair: $(a, b) \longrightarrow a^b$. Hence all possible pairs of (a, b) can be generated from the cross product of A and B. Since A and B are countable, $A \times B$ is also countable. Hence D is countable.

E is also countable for the same reason.

And finally, we can list the elements of the set $D \cup E$ as follows: $\{d_0, e_0, d_1, e_1, d_2, e_2, ...\}$. So the union of D and E is countable. Hence C is countable.

8) For arbitrary strings X, Y and Z, show that $(XYZ)^R = Z^R Y^R X^R$, where, by notation, V^R is the string obtained by reversing the string V.

Let Σ be the alphabet and $X, Y \in \Sigma^*$.

Basis case

Let $X_1 = x_1$ where $x_1 \in \Sigma$, then $X_1^R = x_1$. In other word, any element of Σ is the reverse of itself.

OPTIONAL: To see if there's any pattern, let see what happen when an element from Σ is added at the end of X_1. Let $x_2 \in \Sigma$ and $X_2 = X_1 x_2$, then $X_2^R = (X_1 x_2)^R = (x_1 x_2)^R = x_2 x_1 = x_2 X_1^R$. We can observe that the last letter in a string become the first one in the reserved string.

Induction hypothesis

Let $X_n = Y$, where $Y = (x_1 x_2 \dots x_n) \in \Sigma^*$, then $X_n^R = Y^R = (x_n \dots x_2 x_1)$.

Induction step

Let $X_{n+1} = Y x_{n+1} = (x_1 x_2 \dots x_n) x_{n+1}$, then $X_{n+1}^R = x_{n+1} Y^R = x_{n+1} (x_1 x_2 \dots x_n)^R = x_{n+1}(x_n \dots x_2 x_1)$.

9) For any language A, let A^R be $\{X^R | X \in A\}$. Then, for arbitrary *languages* A and B, explicitly write down the expressions for $(AB)^R$, $(A \cap B)^R$ and $(A \cup B)^R$. Briefly argue why your results are true.

9a) Let $(AB) = \{xy | x \in A, y \in B\}$.

Following from the previous question, $(xy)^R = y'x'$. Therefore:
$(AB)^R = \{y'x' | y' \in B^R, x' \in A^R\} = B^R A^R$

9b) Let $(A \cap B) = \{x | x \in A \wedge x \in B\}$.

We now reverse all the strings in $(A \cap B)$:

$(A \cap B)^R = \{x' | x' \in A^R \wedge x' \in B^R\} = A^R \cap B^R$ where $x' = x^R$.

9c) Let $(A \cup B) = \{x | x \in A \vee x \in B\}$.

We now reverse all the strings in $(A \cup B)$:

$(A \cup B)^R = \{x' | x' \in A^R \vee x' \in B^R\} = A^R \cup B^R$ where $x' = x^R$.

10) If the *languages* A and B are countable, what can you say about the sizes of the above sets?

From the definition of the reverse language, we know that size of a language is equal to the size of its reverse. For each element of $(AB)^R$ can be represented as a pair: $(a, b) \longrightarrow a \cdot b$ where $a \in A$ and $b \in B$. All pairs can be generated by cross product of A and B. Hence $(AB)^R$ is countable.

Each element $x \in (A \cap B)^R$ is also in A^R and in B^R. Since A^R and B^R are countable, the largest possible set is when both sets are equals. This implies that $(A \cap B)^R$ is countable. Similarly, each element $x \in (A \cup B)^R$ is in A^R or B^R. Since A^R and B^R are countable, the largest possible set is when both sets are disjoint, and the size of $(A \cup B)^R$ is the sum of the sizes of A^R and B^R. This implies $(A \cup B)^R$ is countable.