ENGR 213, Fall Semester 2014, Midterm Test 1, Solutions

Problem 1. Solve the given first-order initial value problems:

$$\textbf{(a)} \quad \frac{dy}{dx} = e^{2x}e^{-y} + e^{3x}e^{-y}, \quad y(0) = 0.$$

Solution 1. First, we solve the given differential equation or in other words, we find an one-parameter family of solutions of the given DE. The given differential equation is **separable** so, we separate the variables. Factor out the e^{-y} to obtain:

$$\frac{dy}{dx} = e^{-y}(e^{2x} + e^{3x}) \quad \Rightarrow \quad e^{y}dy = (e^{2x} + e^{3x})dx.$$

Integrating both sides we obtain the solution in an implicit form:

$$\int e^{y}\,dy = \int (e^{2x} + e^{3x})\,dx \quad \Rightarrow \quad \mathbf{e^{y} = \frac{1}{2}e^{2x} + \frac{1}{3}e^{3x} + c}$$

Next, we apply $\ln(\circ)$ at both sides of the implicit solution in order to obtain an explicit solution (one-parameter family of solutions in an explicit form):

$$\ln(e^{y}) = \ln\left(\frac{1}{2}e^{2x} + \frac{1}{3}e^{3x} + c\right) \quad \Rightarrow \quad \mathbf{y = \ln\left(\frac{1}{2}e^{2x} + \frac{1}{3}e^{3x} + c\right)}$$

In order to obtain the solution of the given first-order IVP we apply the given initial value condition. We have two possibilities: To apply the initial value condition in the implicit form of the solution:

$$e^{0} = \frac{1}{2}e^{2(0)} + \frac{1}{3}e^{3(0)} + c \Rightarrow 1 = \frac{1}{2} + \frac{1}{3} + c \Rightarrow c = 1 - \left(\frac{1}{2} + \frac{1}{3}\right) = \frac{1}{6}$$

or to apply the initial value condition to the explicit form of the solution:

$$0 = \ln\left(\frac{1}{2}e^{2(0)} + \frac{1}{3}e^{3(0)} + c\right) \Rightarrow 0 = \ln(5/6 + c) \Rightarrow 1 = 5/6 + c \Rightarrow c = 1/6.$$

Finally, the unique solution of the given IVP is:

$$\mathbf{y = \ln\left(\frac{1}{2}e^{2x} + \frac{1}{3}e^{3x} + \frac{1}{6}\right)}.$$

Solution 2. Being separable, the given d.e. is exact. Representing the given d.e. in a differential form we obtain:

$$(-e^{2x} - e^{3x})dx + e^{y}dy = 0.$$

$$f_x(x,y) = -e^{2x} - e^{3x} \quad \Rightarrow \quad f(x,y) = -\frac{1}{2}e^{2x} - \frac{1}{3}e^{3x} + c(y)$$

and

$$c^{'}(y) = f_y(x,y) = e^y \quad \Rightarrow \quad c(y) = e^y + c.$$

Then

$$f(x,y) = -\frac{1}{2}e^{2x} - \frac{1}{3}e^{3x} + e^y + c$$

and

$$-\frac{1}{2}e^{2x} - \frac{1}{3}e^{3x} + e^y = c$$

is a solution of the given d.e. that can be written also in the form

$$e^y = \frac{1}{2}e^{2x} + \frac{1}{3}e^{3x} + c$$

and in explicit form

$$\mathbf{y} = \ln\left(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{e^{2x}} + \frac{\mathbf{1}}{\mathbf{3}}\mathbf{e^{3x}} + \mathbf{c}\right).$$

As in the first solution applying the initial value condition we obtain the solution of the given IVP:

$$\mathbf{y} = \ln\left(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{e^{2x}} + \frac{\mathbf{1}}{\mathbf{3}}\mathbf{e^{3x}} + \frac{\mathbf{1}}{\mathbf{6}}\right).$$

(b) $\sin(x)\, y^{'} + \cos(x) y = \dfrac{1}{\cos^2(x)}, \quad y\left(\dfrac{\pi}{4}\right) = 0, \quad 0 < x < \dfrac{\pi}{2}.$

Solution 1. The given differential equation is **linear in y.** First, we solve the given differential equation by using **the method of integrating factor.**
Standard form:

$$y^{'} + \frac{\cos(x)}{\sin(x)} y = \frac{1}{\cos^2(x)\sin(x)}.$$

Integrating factor:

$$e^{\int \frac{\cos(x)}{\sin(x)}\, dx} = e^{\ln(|\sin(x)|)} = e^{\ln(\sin(x))} = \sin(x).$$

The method of computing the indefinite integral: $\int \frac{\cos(x)}{\sin(x)}\, dx$ in the integrating factor. Substituting $v = \sin(x),\ dv = \cos(x) dx$ in the above indefinite integral, we obtain:

$$\int \frac{\cos(x)}{\sin(x)}\, dx \ = \int \frac{1}{v} dv = \ln|v| + c = \ln|\sin(x)| + c = \ln(\sin(x)) + c$$

taking into account that $\sin(x) > 0$ for $x \in (0, \pi/2)$. We need only one integrating factor so, we take $c = 0$.
Multiplying the standard form of the differential equation by the integrating factor, we obtain

$$\sin(x) y' + \cos(x) y = \frac{1}{\cos^2(x)} \Rightarrow [\sin(x) y]^{'} = \frac{1}{\cos^2(x)}.$$

Integrating both sides of the above equation gives:

$$\int [\sin(x)\, y]'\, dx = \int \frac{1}{\cos^2(x)}\, dx$$

$$\Rightarrow \ \sin(x)\, y = \tan(x) + c \quad \Rightarrow \quad \sin(x) y = \frac{\sin(x)}{\cos(x)} + c.$$

Solving the last functional equation for $y = y(x)$, we obtain an one-parameter family of solutions of the given differential equation in an explicit form:

$$\mathbf{y} = \frac{\mathbf{1}}{\cos \mathbf{x}} + \frac{\mathbf{c}}{\sin \mathbf{x}}.$$

We apply the given initial value condition in order to solve the given initial value problem:

$$0 = y(\pi/4) \; = \; \frac{1}{\cos(\pi/4)} + \frac{c}{\sin(\pi/4)} \quad \Rightarrow \quad 0 = \frac{2}{\sqrt{2}}(1+c) \quad \Rightarrow \quad \mathbf{c = -1}.$$

The unique solution of the given IVP given in three different equivalent forms:

$$\mathbf{y(x)} = \frac{\mathbf{1}}{\cos \mathbf{x}} - \frac{\mathbf{1}}{\sin(\mathbf{x})}; \quad \mathbf{y(x)} = \frac{\sin(\mathbf{x}) - \cos(\mathbf{x})}{\sin(\mathbf{x})\cos(\mathbf{x})}; \quad \mathbf{y(x)} = \frac{\mathbf{2}(\sin(\mathbf{x}) - \cos(\mathbf{x}))}{\sin(\mathbf{2x})}.$$

Solution 2. The given d.e. is exact, also. Representing the d.e. in a differential form we obtain

$$\left(\cos(x)\, y - \frac{1}{\cos^2(x)}\right) dx \, + \, \sin(x) dy = 0$$

$$M(x,y) = \cos(x)y - \frac{1}{\cos^2(x)}, \quad M_y(x,y) = \cos(x)$$

$$N(x,y) = \sin(x), \quad N_x(x,y) = \cos(x)$$

hence,

$$M_y(x,y) = \cos(x) = N_x(x,y)$$

and we conclude that the given d.e. is exact.

$$f_x(x,y) = M(x,y) = \cos(x)y - \frac{1}{\cos^2(x)} \quad \Rightarrow \quad f(x,y) = \sin(x)y - \tan(x) + c(y)$$

$$\sin(x) + c^{'}(y) \; = \; f)y(x,y) \; = \; \sin(x) \quad \Rightarrow \quad c^{'}(y) = 0 \Rightarrow c(y) = c$$

and in view of this

$$f(x,y) = \sin(x)y - \tan(x) + c \quad \Rightarrow \quad \sin(x)y \, - \tan(x) = c$$

is an implicit solution of the given d.e. Next applying the initial value condition we obtain:

$$\sin(\pi/4)(0) - \tan(\pi/4) = c \quad \Rightarrow \quad c = -1 \quad \Rightarrow \quad \sin(x)y \, - \tan(x) = -1$$

and finally, an explicit solution of the given IVP is:

$$y(x) = \frac{\tan(x)}{\sin(x)} - \frac{1}{\sin(x)} = \frac{1}{\cos(x)} - \frac{1}{\sin(x)}.$$

(c) $(y^2+\cos(y))dx \ + \ (2xy+\ln(y)-x\sin(y))dy = 0, \quad y(0)=1.$

Solution 1. First, in order to check if the DE is exact we have to represent it in a differential form but the given differential equation is in a differential form:

$$(y^2+\cos(y))dx+(2xy+\ln(y)-x\sin(y))dy=0\,.$$

$$M(x,y)=y^2+\cos(y); \quad N(x,y)=2xy+\ln(y)-x\sin(y).$$

Next, we check if the given differential equation is exact:

$$M_y \ = \ 2y-\sin(y) \ = \ N_x,$$

hence, the given differential equation is exact. Then, we know that a function $f(x,y)$ exists such that:

$$\begin{aligned} f_x &= y^2+\cos(y) \\ f_y &= 2xy+\ln y - x\sin(y)\,. \end{aligned}$$

and we solve the above system of partial differential equations in order to find $f(x,y)$. We integrate $f_x = y^2+\cos(y)$ in x and evaluate $h(y)$ in order to find $f(x,y)$:

$$\begin{aligned} & f(x,y) = y^2x + x\cos(y) + h(y) \\ & 2xy+\ln y - x\sin(y) = f_y = 2yx - x\sin(y) + h'(y) \\ & 2xy+\ln(y)-x\sin(y) = 2yx - x\sin(y) + h'(y) \quad \Rightarrow h'(y)=\ln(y) \\ & \int h^{'}(y)dy = \int \ln(y)\,dy \text{ (integration by parts)} \\ & h(y) = y\ln(y) - y + c \\ & f(x,y) = y^2x + x\cos(y) + y\ln(y) - y + c\,. \end{aligned}$$

Hence, an implicit one-parameter family of solutions of the given differential equation is:

$$y^2x + x\cos(y) + y\ln(y) - y = c.$$

Solving for $x=x(y)$ we obtain an explicit solution:

$$x(y^2+\cos(y)) = y - y\ln(y) + c \Rightarrow \mathbf{x} = \frac{\mathbf{y}-\mathbf{y}\ln(\mathbf{y})+\mathbf{c}}{\mathbf{y^2}+\cos(\mathbf{y})}.$$

Now we apply the given initial value condition **either in the implicit form or in the explicit form** in order to solve the given initial value problem.
(1) In the implicit form:

$$y(0) = 1,\ x(1) = 0,\ (x = 0,\ y = 1) \Rightarrow (1)^2(0) + (0)\cos(1) + 1\ln(1) - 1 = c \Rightarrow \ \mathbf{c = -1}$$

and in view of this first we obtain an implicit solution of the initial value problem and solving for x we obtain an explicit solution:

$$y^2x + x\cos(y) + y\ln(y) - y = -1 \Rightarrow \mathbf{x} = \frac{\mathbf{y - y\ln(y) - 1}}{\mathbf{y^2 + \cos(y)}}.$$

(2) In the explicit form: $y(0) = 1,\ x(1) = 0,\ (x = 0,\ y = 1)$

$$0 = \frac{1 - (1)\ln(1) + c}{(1)^2 + \cos(1)} \Rightarrow 0 = \frac{c + 1}{1 + \cos(1)} \Rightarrow \mathbf{c = -1}$$

$$\Rightarrow \mathbf{x(y)} = \frac{\mathbf{y - y\ln(y) - 1}}{\mathbf{y^2 + \cos(y)}}.$$

Solution 2. The given d.e. is linear in x:

$$(y^2 + \cos(y))\frac{dx}{dy} + (2y - \sin(y))x = -\ln(y).$$

Standard form by using Newton's notation of derivative:

$$x' + \frac{2y - \sin(y)}{y^2 + \cos(y)}x = -\frac{\ln(y)}{y^2 + \cos(y)}.$$

$$p(y) = \frac{2y - \sin(y)}{y^2 + \cos(y)}, \quad \int \frac{2y - \sin(y)}{y^2 + \cos(y)}dy = \ln(y^2 + \cos(y))$$

and the intergrating factor is

$$e^{\int p(y)dy} = e^{\ln(y^2 + \cos(y))} = y^2 + \cos(y).$$

Then multiplying the standard form by the integrating factor we obtain

$$\Big[(y^2 + \cos(y))x(y)\Big]' = -\ln(y)$$

and after integrating both sides we obtain

$$(y^2 + \cos(y))x(y) = y - y\ln(y) + c\,.$$

Applying the given initial value condition $x = 0, y = 1$ we obtain

$$1^2 + \cos(1)(0) = 1 - 1\ln(1) + c \quad \Rightarrow \quad c = -1$$

and solving for x we obtain an explicit in $x = x(y)$ solution:

$$\mathbf{x(y)} = \frac{\mathbf{y} - \mathbf{y}\ln(\mathbf{y}) - \mathbf{1}}{\mathbf{y^2} + \cos(\mathbf{y})}.$$

Problem 2. Solve the given first-order differential equations by using appropriate substitutions:

$$\textbf{(a)} \quad y^{'} \; = \; \frac{y^2 + x\,y}{x^2}\;;$$
$$\textbf{(b)} \quad y^{'} = 2 \; + \; (y - 2x + 1)^2\,.$$

Solution 1 of (a).

$$\frac{dy}{dx} = \frac{y^2 + x\,y}{x^2} \Rightarrow x^2\,dy \; - (y^2 + x\,y)dx = 0$$

$$M(x, y) = -(y^2 + xy);\;\; M(tx, ty) = -((ty)^2 + (tx)(ty)) = -t^2(y^2 + xy) = t^2 M(x, y)$$
$$N(x, y) = x^2;\;\; N(tx, ty) = (tx)^2 = t^2 x^2 = t^2 N(x, y)$$

and in view of this M and N are homogeneous of the same order 2. Then

$$x^2 y^{'} - (y^2 + xy) \quad \Rightarrow \quad x^2 \left[y^{'} - \left(\left(\frac{y}{x}\right)^2 + \frac{y}{x} \right) \right] = 0 \quad \Rightarrow \quad y^{'} - \left(\left(\frac{y}{x}\right)^2 + \frac{y}{x} \right) = 0.$$

Substituting

$$\frac{y}{x} = v;\;\; y = xv;\;\; (y = y(x),\;\; v = v(x)); y^{'} = v + xv^{'}$$

we obtain a differential equation in v and x":

$$(v + xv^{'}) - (v^2 + v) = 0 \quad \Rightarrow \quad xv^{'} - v^2 = 0$$

Obviously, the differential equation in v and x is separable:

$$x\frac{dv}{dx} = v^2 \quad \Rightarrow \quad \frac{dv}{v^2} = \frac{dx}{x}$$

$$\int \frac{dv}{v^2} = \int \frac{dx}{x} \quad \Rightarrow \quad -\frac{1}{v} = \ln|x| + c$$

$$\Rightarrow \frac{y}{x} = v = -\frac{1}{\ln|x| + c} \quad \Rightarrow \quad \mathbf{y} = -\frac{\mathbf{x}}{\ln|\mathbf{x}| + \mathbf{c}}.$$

Solution 2 of (a).

$$y^{'} = \frac{y^2 + x\,y}{x^2} \Rightarrow x^2\,y^{'} \; - \; xy = y^2$$

and this is Bernoulli's differential equation with $n = 2$. Then

$$x^2 y' - xy = y^2 \quad \Rightarrow \quad \mathbf{x^2 y^{-2} y' - x y^{-1} = 1}$$

Bernoulli's differential equation with $n = 2$.
Method 1 by using the ready substitution. Substitution

$$u = y^{1-n} = y^{1-2} = y^{-1};\ u' = -y^{-2}y';\ y' = -y^2 u'.$$

Then

$$x^2 y^{-2}(-y^2 u') - xu = 1$$
$$\Rightarrow \quad -x^2 u' - xu = 1$$

that is linear in u first-order differential equation:
Standard form:

$$u' + \frac{1}{x}u = -\frac{1}{x^2}\,.$$

Integrating factor:

$$p(x) = \frac{1}{x};\ e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x.$$

Then

$$xu' + u = -\frac{1}{x} \Rightarrow [xu]' = -\frac{1}{x} \Rightarrow \int [xu]' dx = -\int \frac{1}{x} dx$$

$$\Rightarrow \ xu = -\ln|x| + c \ \Rightarrow \Rightarrow y^{-1} = u = -\frac{\ln|x| + c}{x} \quad \Rightarrow y = -\frac{x}{\ln|x| + c}.$$

Method 2. By division of the both sides of the Bernoulli's differential equation

$$a_1(x)y' + a_0(x)y = f(x)y^n$$

to y^n we remove the dependence on y in the right-hand side:

$$a_1(x)y^{-n}y' + a_0(x)\mathbf{y^{1-n}} = f(x).$$

Then we substitute the expression in y that is a factor of $a_0(x)$ in our case $n = 2$ and $u = y^{1-n} = y^{-1}$:

$$x^2 y' - xy = y^2 \quad \Rightarrow \quad x^2 y^{-2} y' - x\mathbf{y^{-1}} = 1.$$

We substitute $u = y^{-1}$; $u' = -y^{-2}y'$; $y' = -y^2 u'$ and the other steps are the same as in the Method 1.

Solution of (b).

$$y^{'} = 2 \,+\, (y-2x+1)^2; \quad v = y-2x+1,\; y=y(x),\; v=v(x),\; y^{'} = v^{'}+2$$

$$\Rightarrow \quad v^{'}+2 \,=\, 2+v^2 \quad \Rightarrow \quad \frac{dv}{dx} = v^2.$$

The above differential equation is separable:

$$\frac{dv}{v^2} = dx \quad \Rightarrow \quad \int \frac{dv}{v^2} = \int dx$$

$$-\frac{1}{v} = x+c \quad \Rightarrow \quad v = -\frac{1}{x+c}$$

$$\Rightarrow \quad y-2x+1 = -\frac{1}{x+c} \quad \Rightarrow \quad \mathbf{y = 2x - 1 - \frac{1}{x+c}}.$$

Problem 3. A mixing tank holds initially 200 gallons of water in which 20 pounds of salt has been dissolved. Pure water is pumped into the tank at a rate of 4 gal/min and when the solution is well mixed, it is pumped out at the same rate.
(a) Determine an initial value problem governing the dynamics of the amount $A(t)$ of salt in the tank, where t is the time.
(b) Solve the initial value problem obtained in **(a)**.
(c) At what time is the amount of salt in the tank 10 pounds?

Solution. (a) We compute the input rate of salt entering into the tank taking into account that the water pumped in the tank is pure (no salt in it):

$$R_{in} = (4\ gal/min) \times (0\ pound/gal) = 0\ pound/min.$$

Suppose that the amount of salt in the tank at time t is $A(t)$ pounds. Then at time t, the amount of salt in one gallon (the concentration) is $A(t)/200$ so, the output rate of salt leaving the tank is:

$$R_{out} = \left(\frac{A(t)}{200}\ pound/gal\right) \times (4\ gal/min) = \frac{A(t)}{50}\ pound/min.$$

From Calculus the rate of change of the amount of salt in the tank is $A^{'}(t)$ but on the other side the rate of change of salt in the tank is also equal to $R_{in} - R_{out}$. From here:

$$\frac{dA}{dt} = R_{in} - R_{out} \quad \Leftrightarrow \quad A^{'} = R_{in} - R_{out}$$

and replacing R_{in} and R_{out} with their explicit expressions we obtain the DE:

$$\frac{dA}{dt} = 0 - \frac{A}{50} \quad \Rightarrow \quad \frac{dA}{dt} = -\frac{A}{50}.$$

Taking into account that $A(0) = 20$; the first-order linear IVP giving a response of the dynamics of mixing is:

$$\frac{dA}{dt} = -\frac{A}{50} \qquad \textbf{differential equation}$$
$$A(0) = 20 \qquad \textbf{initial value condition}$$

The DE in the IVP is linear but note that it is separable, also. Mathematical models with linear equations are called Linear Models.

(b) We solve the IVP to obtain an explicit formula for $A(t)$. First we find the one-parameter family of all solutions of the differential equation by applying **the method of separating the variables**:

$$\frac{dA}{dt} = -\frac{A}{50} \quad \Rightarrow \quad \frac{dA}{A} = -\frac{1}{50}dt \Rightarrow \int \frac{dA}{A} = \int (-0.02)\, dt$$

$$\Rightarrow \quad \ln|A| = -0.02\,t + c \quad \Rightarrow \quad |A| = e^{(-0.02)t+c} = (e^c)e^{(-0.02)t}$$

$$\Rightarrow \quad |A| = c\,e^{(-0.02)t} \;\; (e^c \;\to\; c > 0) \quad \Rightarrow \quad A = ce^{(-0.02)t} \;\; (c \text{ arbitrary.}$$

The general solution of the differential equation is

$$\mathbf{A(t) = c\,e^{(-0.02)t}} \quad c \;\text{ arbitrary constant.}$$

Now, we use the initial-value condition in order to determine the constant c.

$$20 = A(0) = c\,e^{(-0.02)(0)} = c \quad \Rightarrow \quad c = 20$$

and in view of this the unique solution of the IVP is:

$$\mathbf{A(t) = 20\,e^{(-0.02)t}}.$$

Hence, the amount $A(t)$ of salt in the tank at time t is

$$\mathbf{A(t) = 20\,e^{(-0.02)t}}.$$

(c) At what time is the amount of salt in the tank 10 pounds?

$$10 = 20\,e^{(-0.02)t} \Rightarrow \ln(1/2) = (-0.02)\,t \Rightarrow t = \frac{\ln(0.5)}{-0.02} = 34.657 \text{ min} = 34 \;\, min \;\, 39 \;\, sec\,.$$

Problem 4. A thermometer is removed from a room and it is taken outside, where the temperature is 10° C. One minute after the moment when it was taken outside the thermometer reads $20°C$. Two minutes after the moment when the thermometer was taken outside it reads $18°C$. What is the temperature of the room?

Solution. We have a process of cooling. Let $t = 0$ be the moment when the thermometer is removed from the room and taken outside . Then, $T_0 = T(0)$ is the temperature of the room, $T_m = 10°$ C, $T(1) = 20°$ C, $T(2) = 18°$ C. The initial temperature T_0 is the temperature of the room that we are looking for. Obviously, $T_0 > T_m$ and from here we see that we have a cooling process. According to Newton's law of cooling:

$$T(t) = (T_0 - T_m)e^{kt} + T_m \qquad (k < 0)\,.$$

We want to find T_0 that is the temperature of the room. In order to do this we shall use the two measurements at $t = 1$ min and $t = 2$ min:

$$T(1) = 20°\ C \quad \Rightarrow \quad 20 = (T_0 - 10)e^{k} + 10 \qquad (t = 1)$$
$$T(2) = 18°\ C \quad \Rightarrow \quad 18 = (T_0 - 10)e^{2k} + 10 \qquad (t = 2).$$

We have a system of 2 equations with unknowns T_0 and k. We solve the system in order to find T_0:

$$10 = (T_0 - 10)e^{k}, \qquad 8 = (T_0 - 10)e^{2k}$$
$$\Rightarrow \quad \left(\frac{10}{T_0 - 10}\right)^2 = e^{2k} = \frac{8}{T_0 - 10}$$
$$\Rightarrow \quad \frac{100}{(T_0 - 10)^2} = \frac{8}{T_0 - 10}$$
$$\Rightarrow \quad T_0 - 10 = \frac{100}{8} = 12.5 \quad \Rightarrow \quad \mathbf{T_0 = 22.5°\ C}.$$