The Time Value Of Money

Learning Objectives

When you have finished studying the material in this lesson, you should be able to:

1. Understand the fundamentals of the time value of money, including present and future value concepts both for single cash flows and annuities.

2. Understand the concept of rate of rate of return and the different ways in which it is expressed.

3. Understand the different factors which affect the rate of return.

Lesson Overview

This is one of the most important lessons in the course -- the concept of the time value of money is fundamental to understanding most business and personal finance issues.

Few of us are fortunate enough to currently have sufficient financial resources to meet all of our future financial goals -- therefore, to meet these future commitments, funds will have to be saved over time so that they accumulate to the required levels.

Several questions arise, however, in trying to meet such financial goals. How much must I earn on invested funds so that they accumulate to the desired level? How much do I have to save each year if I can earn say 8% on invested funds to be able to meet the required level? How much will I have to accumulate by retirement so that I can spend say $50,000 throughout my retirement years as well as being able to leave $500,000 to my heirs?

It is answers to questions like these that this lesson prepares you to answer.

Compounding and Discounting

The fundamental relationship underlying the time value of money concept is given in Figure 1: if you have an amount today that will be invested for a number of years, it will become a **future value**, and the mathematical process is called ***compounding***.

In some cases, you know how much will be received in the future, for example, a bond will make fixed coupon payments over its life and repayment of principal at maturity, and the question arises as to what is the appropriate current value for such a bond. The mathematical process that brings these future cash flows to obtain their **present value** is called ***discounting***.

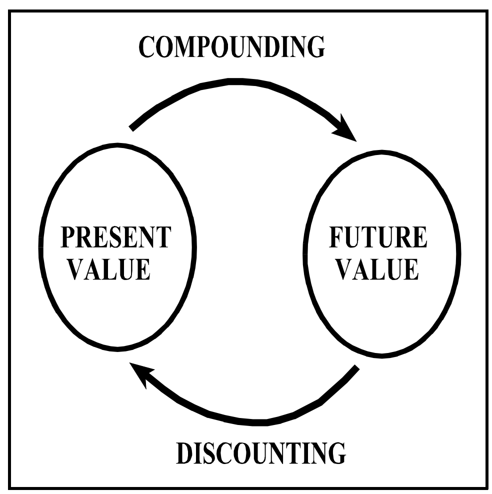


Figure 1 Basic Structure of Time Value of Money Problems

Regardless of the terminology involved, the mathematical processes are just "flip sides" of one another, as Figure 1 and the remainder of this lesson indicates.

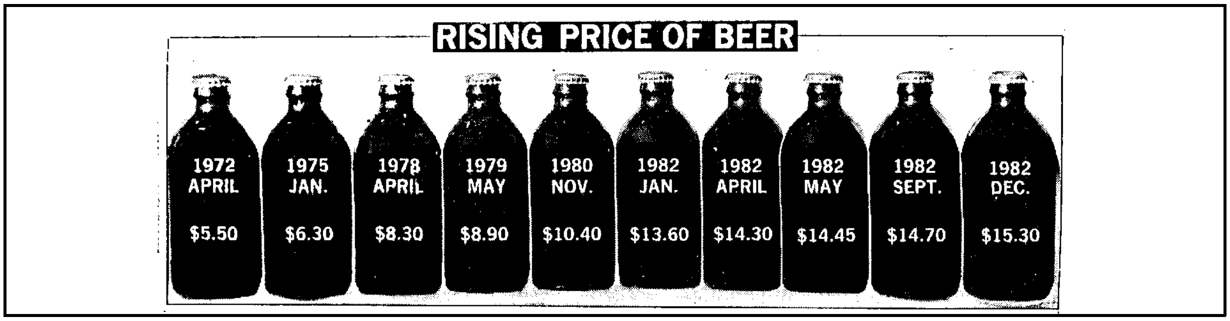


Figure 2 The Price of a 24-Bottle Case of Beer, including Bottle Deposit

A. The Future Value of a Single Amount

Deposit the price of a case of beer in January 1975 ($6.30) in a bank account that pays 8% per annum, compounded annually.

Would you have earned enough money in the bank account by January 1978 to purchase a case of beer ($8.30/case)*, without additional cash inflows?*

Solution:

Calculate the future value of $6.30, compounded at 8% per annum, for three years.

If you update your bank passbook annually, your annual balance would appear as follows:

	Item	**Amounts**	**Balance**
Jan 1975	Deposit	6.30	6.30
Jan 1976	Interest	0.504 { 8% x 6.30	6.804 [A]
Jan 1977	Interest	0.544 { 8% x 6.804	7.348 [B]
Jan 1978	Interest	0.588 { 8% x 7.348	7.936

[A] 50.4 cents interest was earned during 1975 (8% x 6.30), which when added to the original principal of 6.30 produces 6.804

[B] 54.4 cents interest was earned during 1976 (8% x 6.804), which when added to the principal at the start of the year (6.804) produces 7.348

The Future Value of $6.30, compounded at 8% per annum for three years is only <u>$7.936</u>, so to answer the question, no, you would not have enough funds available to buy a case of beer three years later by simply earning 8% per annum on your money. (Note that this tells you something about the rate of increase in beer prices over this period of time -- more on this topic later).

General Equation for the Future Value of a Single Payment

$$F = P \times (1 + i)^{n}$$

where,

F	Future value of the original investment
P	the original investment (or Present Value)
i	interest rate for each period
and, n	number of periods of compounding

Example:

The amount that is accumulated in the account at the **end of two years**, in the above

$$6.30 \xrightarrow{1.08} 6.804 \xrightarrow{1.08} 7.348$$

problem,
is more compactly calculated as

$$6.30 \times (1.08)^{2} = \underline{\underline{7.348}}$$

The future value factor could be obtained from the table in Appendix 1-B-1 (p.43) { ***Future-Value (Compounded Sum) of $1 After a Given Number of Time Periods***)}

Period	7%	8%	9%
2	1.1449	**1.1664**	1.1881

to produce the same results, namely, 6.30 x 1.1664 = **7.348**

Exercise:

Show that if Jean Pantz deposits $2,000 in a bank account that earns 8% per annum, compounded annually, after ten years she will have $4,317.85 on deposit.

B. Changing the Compounding Period:

Consider the growth of the above account at 8% per annum with ***semi-annual compounding***:

	Item	Amount	Balance
Jan 1975	Deposit	6.30	6.30
Jul 1975	Interest	0.252 { 4% x 6.30	6.552
Jan 1976	Interest	0.262 { 4% x 6.552	6.814
etc....			
Jan 1978	Interest	0.306 { 4% x 7.665	**7.972**

If you compare the final balance under the two compounding methods,

Semi-Annual:	\$7.972
Annual	\$7.936

one concludes that ***more frequent compounding results in larger terminal values***.

Notice that the effect of semi-annual compounding has been to put interest into the account faster, and that interest earns additional interest faster..... The overall amount accumulated may not seem to be a big deal here given that the two accumulated amounts differ by only a few pennies. However, as i) the dollar amount invested, and/or ii) the number of compounding periods increase, and/or iii) the interest rate increases, the difference between the accumulated amounts will become much more significant.

The **future value formula** for multiple compounding periods is:

$$\mathbf{F = P \times (1 + i/n)^{t \times n}}$$

where,

n number of compounding periods per year

t number of years in the investment horizon

and,

t x n total number of compounding periods in the investment horizon.

By formula then,

$$\begin{aligned} F &= \$6.30 \times (1 + .08/2)^{3 \times 2} \\ &= \$6.30 \times 1.265 \\ &= \underline{\underline{\mathbf{\$7.972}}} \end{aligned}$$

The **f**uture **v**alue **i**nterest **f**actor (**FVIF**) could be obtained from the table in Appendix 1-B-1 (p.43)

Period	3%	4%	5%
6	1.1941	**1.2653**	1.3401

For semi-annual compounding, use one half of the interest rate, but twice the number of periods to obtain the appropriate future value factor.

Exercises:

1. *Verify the following accumulated amounts when an initial deposit of $6.30 is invested for 3 years at 8% per annum, with the following compounding frequency*:

Compounding Frequency	Accumulated Amount
Annual	$7.936
Semi-Annual	$7.972
Quarterly	$7.98
Monthly	$8.002
Daily	$8.0087

2. *Verify the following accumulated amounts when an initial deposit of $2,000 is invested for 10 years at 8% per annum, with the following compounding frequency*:

Compounding Frequency	Accumulated Amount
Annual	$4,317.85
Semi-annual	4,382.25
Quarterly	4,416.08
Monthly	4,439.28
Daily	4,450.69

C. The Present Value of a Single Amount

As indicated in Figure 1, the Present Value problem is really the flip side of a future value problem, and this is illustrated mathematically below.

Future Value Calculations: $F = P \times (1 + i)^{n}$

Present Value Problems: $P = \dfrac{F}{(1 + i)^{n}}$

for multiple compounding periods: $P = \dfrac{F}{(1 + i/n)^{n \times t}}$

Present Value Example:

If you had a negotiable note which guaranteed that you would receive $79.36 payable in 3 years, and you could invest at 8% compounded annually, would you be willing to accept $68.00 for it today?

Present Value from the above formula:

$$P = \frac{F}{(1+i)^n} = \frac{\$79.36}{(1+.08)^3} = \underline{\underline{\$63.00}}$$

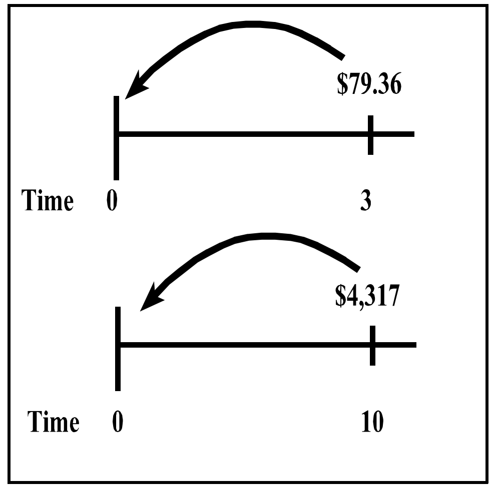


Figure 1 Use of Time Lines in Present Value Problems

Would you accept $68.00 for the note? Yes, since the (present) value of the note is only $63 to you.

Using the Present Value Tables:

The **p**resent **v**alue **i**nterest **f**actor for a single cash flow (**PVIF**) could be obtained from the table in Appendix 1-B-3 (p.45) { ***Present Value of $1 Received at the End of a Given Number of Time Periods*** }

Period	7%	8%	9%
3	0.8163	**0.7938**	0.7722

The present value factor 0.7938 simply comes from ***discounting*** $1 for 3 periods at 8%,

$$\frac{\$1.00}{(1+.08)^3} = \underline{\underline{0.7938}}$$

$$P = \$79.36 \times 0.7938 = \underline{\underline{\$63.00}}$$

To find the present value of $79.36, just multiply the present value factor for $1 by $79.36 to produce the result of $63 as indicated above.

Present and future value problems can become quite complicated: to help you understand exactly what is happening to the cash flows in the problem, you should prepare a **time line**, as indicated in Figure 3 for the above problem, and the following exercise.

Exercise:

You are guaranteed a payment of $4,317 in ten years from a negotiable note. Show that if your discount rate was 8% per annum, with annual compounding, the minimum price that you would be willing to sell the note for would be $2,000.

D. Determining the Yield or Return on a Simple Investment

What was the rate of growth of beer prices between January 1975 and January 1978 (still assuming that the January 1978 and April prices are the same).

PV = \$6.30 FV = \$8.30 Time Span = 3 years

only the yield term, i, is unknown.

Solving for this interest rate or yield, or in this case, the growth rate, i:

$$P = \frac{F}{(1+i)^n}$$

$$6.30 = \frac{\$8.30}{(1+i)^3}$$

How should one solve for i?
There are several ways available:

1. Financial calculator
2. Present (or future) value tables
3. Trial and error, with some linear interpolation:

Solving for i by Trial & Error, followed by linear interpolation:

Interest Rate	Future Value	Guess Number	Next Guess
8%	\$7.936	1	{ higher
9	\$8.158	2	{ higher
?%	☞ **\$8.30**		**{ the answer**
10	\$8.385	3	{ lower

Clearly, the correct rate of growth is closer to 10% than to 9%, but the question remains as to exactly where in the 9% to 10% range is the correct answer.

The following table describes the problem:

Interest Rate	**9%**	**i%**	**10%**
Future Value	\$8.158	\$8.30	\$8.385

Linear interpolation of the 9 to 10% range yields **i = 9.63%,** as follows:

$$\frac{10 - \mathbf{i}}{10 - 9} = \frac{8.385 - 8.30}{8.385 - 8.158}$$

Students often have difficulty remembering how to set up such an interpolation -- it is really quite simple, however: you are trying to determine what proportion of the distance between 9% and 10% does the interest rate i fall to give a future value of 8.30.

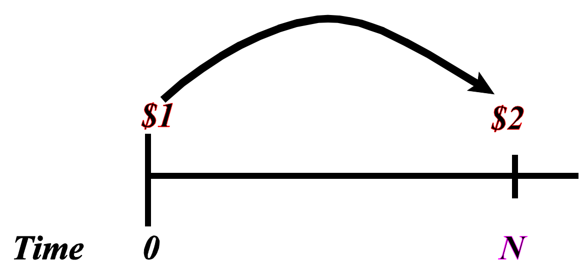
The easiest way to set up the proportion is to make the left hand side of the proportion (both numerator and denominator) **positive**: since i is greater than 9 but less than 10, then 10-i will be positive, and clearly 10-9 is also positive so that the left hand side of the proportion is positive.

Then it is simply a matter of copying the corresponding future value terms to create the right hand side of the proportion: e.g., corresponding to 10%, use 8.385, for i%, use 8.30, and finally for 9%, use 8.158. You are left with 1 equation, in one unknown, i, which can be easily solved.

E. Determine the Time Required to Accumulate a Future Value

If interest rates are 7%, how long does it take for money to double in value?

This is simply a present (or future) value problem, in which the starting and ending values are known. Since the interest rate is given, all that has to be determined is the time required for the conversion, of say $1 to $2 (or $5,000 to $10,000).



Solve for N in the following equation:

$1 x **FVIF**(N years, 7%) = $2

or $\$1 \times (1 + i)^{n} = \$2$ ☞ $\$1 \times (1 + 0.07)^{n} = \$2$ ☞ n = <u>**10.2 years**</u>

On a financial calculator, this is an easy task, but manually can be done by linear interpolation across the time range, in a way similar to how interpolation across the interest rate range was accomplished when solving for yield.

Years	**Future Value**	**Guess Number**	**Next Guess**
9	$1.84	1	{ longer
10	$1.97	2	{ longer
?n	☞ **$2.00**		**{ the answer**
11	$2.10	3	{ shorter

Years	**10**	**n**	**11**
Future Value	$1.97	$2.00	$2.10

Linear interpolation of the 10 year to 11 year range yields **n = 10.2 years** as follows:

$$\frac{11 - \mathbf{n}}{11 - 10} = \frac{2.10 - \mathbf{2.00}}{2.10 - 1.97}$$

Exercise:

Show that it takes 16.2 years to triple your money at 7% p.a.

F. Conversion from One Compounding Frequency to Another

The **APR**, or ***annual percentage rate***, is the yearly interest rate quoted on a loan (or investment). It is often referred to as the **nominal rate** on the loan.

The APR is determined by multiplying the stated interest rate per compound period by the number of compound periods per year.

APR = stated rate per compound period x number of compound periods per year

For example, the APR for a loan quoted as costing 3% per quarter will be

APR = 3%/quarter x 4 quarters per year = 12% per annum

Note that the APR does not take compounding into account.

The effective annual interest rate will take compounding into account, and will determine the true cost of such a loan.

$$\$1 \times (1 + 0.03)^{4} = 1.125$$

So that the effective annual interest rate is 1.125 - 1 = 12.5%

The formula which allows for this conversion is

$$\mathbf{(1 + R) = (1 + i/n)^{n}}$$

where
R is the effective annual interest rate
i is the nominal interest rate
n is the number of compound periods per year

For the above example,

$$(1 + R) = (1 + 0.12/4)^{4}$$

from which R = **12.5%**

This also illustrates that the effective annual interest rate is ***the rate at which $1 would actually grow over 1 year, if there was only 1 compounding period per year***.

The concept is similar to that which will be shown below where there are multiple periods in the investment horizon and interest rates changes over the time frame; we will be able to calculate a single rate which produces the same terminal result as was actually achieved through the actual series of rate changes.

This concept of effective annual interest also allows us to convert from one compounding frequency to another.

For example, *Confederation Bank* pays interest on its deposits at the rate of 6% per annum, compounded annually. *The Red River Bank* quotes a deposit rate which is compounded daily, while *The Northern Lights Bank* quotes a deposit rate which is compounded semi-annually. *What must be the effective daily and effective semi-annual rates being quoted by these two latter banks if all of the banks end up paying the same effective annual rate? What is the APR under each of these cases?*

For *Confederation Bank*:

APR = 6% = Effective annual Rate

Basically, $1 becomes $1.06 by the end of the year

For *The Red River Bank*:

$$(1.06) = (1 + d)^{365}$$

where d is the daily rate = **0.01596% per day**

So that the APR would be 0.01596% x 365 = **5.827%**

For *The Northern Lights Bank*:

$(1.06) = (1 + s)^2$

where s is the semi-annual rate = **2.956% per half year**

So that the APR would be 2.956% x 2 = **5.91%**

Exercise:

Show that if the effective annual rate was 6% per annum, if compounding were to occur quarterly or weekly the effective weekly and effective quarterly rates would be 0.112% per week and 1.467%, respectively. Also show that the APR under each of those scenarios would be 5.83% p.a. and 5.86% p.a., respectively.

G. Multi-Period Rates of Return

Assume that $6.30 is deposited into a savings account, but interest rates change as follows:

Year	Interest Rate
1	8%
2	9
3	10
4	11

i) *How much will be on deposit at the end of 4 years ?*

$$6.30 \xrightarrow{1.08} 6.804 \xrightarrow{1.09} 7.416 \xrightarrow{1.10} 8.158 \xrightarrow{1.11} 9.055$$

This can be written more compactly as:

$$6.30 \times (1.08) \times (1.09) \times (1.10) \times (1.11) = \underline{\underline{\mathbf{9.055}}}$$

What is the effective annual rate of return on this four year investment? i.e., what single rate of return, earned each and every year, would produce the same terminal value as is earned from the above series of interest rates ?

$$P = \frac{F}{(1+i)^n}$$

$$6.30 = \frac{9.055}{(1+i)^4}$$

Solving for r, yields r = $\underline{\underline{\mathbf{9.49\%}}}$ per annum

Note that never did the investor earn 9.49% in any year, but the overall effect is "***as though***" that was the amount that was earned each year. An alternative way to calculate the effective annual rate is as follows:

$$(1.08 \times 1.09 \times 1.10 \times 1.11)^{1/4} - 1 = \underline{\underline{\mathbf{9.49\%}}}$$

This approach is also referred to as the ***geometric mean return*** of the series of interest rates.

- The ***geometric return*** is calculated by adding 1 to each periodic return, multiplying these values together, taking the nth root of the product and subtracting 1. It is also called the ***constant rate of return*** because an investment compounding at the geometric return will grow to the same value that the actual returns produced.

H. Multiple Cash Flows Over Time

There are two cases of multiple cash flows over time to be considered:

1) ***irregular cash flows*** over time: some cash flows may be the same, some may differ.
2) ***regular cash flows*** that are the same over time. In this latter category, there are two types of cash flow streams: i) ***annuities***, in which the cash flows persist only for a limited time, and ii) ***perpetuities***, in which the cash flows continue indefinitely.

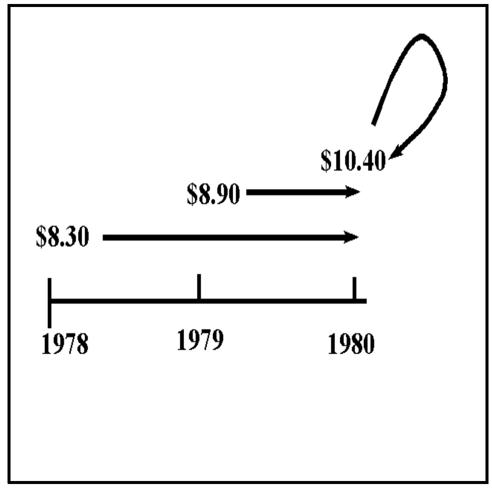


Figure 0 Irregular Cash Flows Over Time

Case 1: Irregular Cash Flows Over Time:

A person deposits the following into his bank account at the start of each of the following years:

Year	Deposit
1978	8.30
1979	8.90
1980	10.40

What amount will have accumulated at the start of 1980, if the person can earn 8% per annum each year on the funds invested (assume annual compounding)?

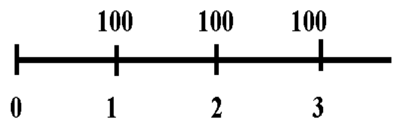
As Figure 4 indicates, this is a future value problem, with an amount accumulating over the two year time frame. The last cash flow is simultaneously added to an removed from the account in 1980, and therefore does not earn any interest.

The future value of the investments has to be worked out separately for each investment, as follows:

			Future Value
$8.30 \times (1.08)^2$	= 8.30 x 1.166	=	9.38
$8.90 \times (1.08)^1$	= 8.90 x 1.08	=	9.61
$10.40 \times (1.08)^0$	= 10.40 x 1.00	=	10.40
Future Value of All Cash Flows			**$29.69**

What is the Present Value at the start of 1978 of the future bank deposits, if the person plans to make the same series of deposits as stated above?

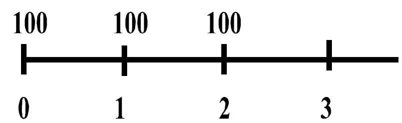
$$\text{Present Value} = \frac{8.30}{(1.08)^0} + \frac{8.90}{(1.08)^1} + \frac{10.40}{(1.08)^2}$$
$$= 8.30 + 8.24 + 8.92$$
$$= \underline{\underline{\$25.46}}$$



An equivalent method, to achieve the same answer would be to simply present value the future value of all of the cash flows, which was already determined:

$$\text{Present Value} = \frac{29.69}{(1.08)^2} \quad \text{☞ FV of all cash flows}$$
$$= \underline{\underline{\$25.46}}$$

Case 2: Regular Cash Flows Over Time:



i) Annuities

Annuities are defined as a finite series of equal cash receipts or payments.

There are two types of annuities:

1) ***Ordinary Annuity*** or ***An Annuity in Arrears***

In this type of annuity, payments are received or made at the ***end of each period***, as the time line to the right indicates for a 3 year annuity of $100 per year.

2) ***An Annuity Due*** or ***An Annuity in Advance***

In this type of annuity, payments are received or made ***at the start of each period***, as the time line to the right indicates: this also is a 3 year annuity of $100 per year, but the first cash flow begins ***immediately***.

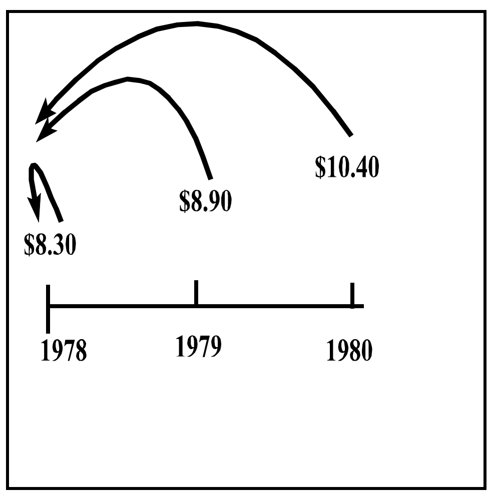


Figure 1 Present Value of Irregular Cash Flows

You need to be aware that the annuity tables in the text(in fact, most texts) and older financial calculators are constructed only for ordinary annuities. Most of the newer financial calculators have a toggle switch or button that allows the user to switch between one form and the other -- the user must take care, however, that the correct form is being used.

It is actually quite easy to determine the present or future value of an annuity due from regular annuity tables, however.

The Present Value of an annuity due can be thought of as coming from :
1) an immediate payment of $100,
and
2) a regular annuity for two years.

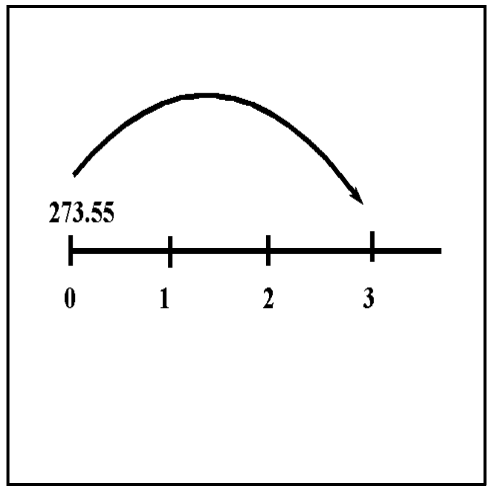


Figure 3 Indirect Determination of the Future Value of an Annuity Due

The regular annuity present value tables can then be used directly for this latter part.

Assuming that the appropriate discount rate is 10% per annum for the above stream, the

Present Value = 100 + 100 x **PVIFA**(2 y, 10%)
= 100 + 100 x 1.736
= **273.55**

where **PVIFA**(2 y, 10%) means the **p**resent **v**alue **i**nterest **f**actor for an **a**nnuity. This can be found in the text from the table in Appendix 1-B-3 (p.45) (This factor is distinguished from the present value interest factor of a single cash flow (**PVIF**) by having an A at the end -- similarly, the future value factor for an ordinary annuity is referred to as **FVIFA**).

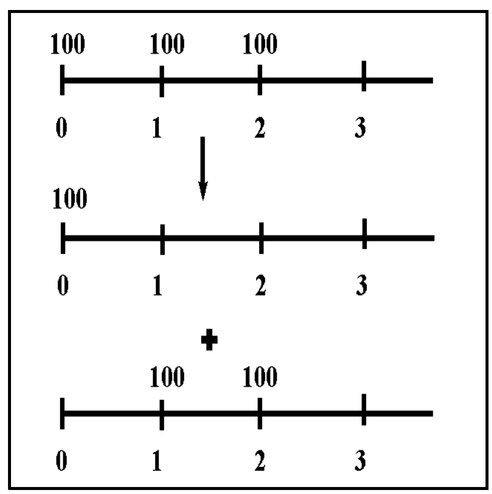


Figure 2 Using Regular Annuity PV Tables for an Annuity Due

The Future value for this stream at year three can be easily found, again by recognizing that present and future value problems are just flip sides of one another: since we have just determined the present value above, simply future value the present value for 3 years at 10% p.a.

Future Value = 273.55 x **FVIF**(3 y, 10%)
= 273.55 x 1.331
= **364.09**

An alternative approach to find the future value of this annuity is to use the future value annuity factor for a regular annuity of 3 years (Appendix 1-B-2 (p.44) and multiply the result by (1 + i)

Future Value = 100 x **FVIFA**(3y, 10%) x (**1.10**) = **364.09**

The Ordinary Annuity Formula

The formula for the present value of an ordinary annuity is $PVIFA = \dfrac{1 - \left[\dfrac{1}{(1+i)^n}\right]}{i}$

This formula may look a little intimidating, but it is really quite easy to use. For example, if you want to determine the present value factor for a 3 year (regular) annuity of $1 when the discount rate is 10%, then

To determine the **present value of an annuity in advance**, again use the present value factor for a regular annuity and multiply the result by (1 + i).

For example, for the 3 year annuity of $100 in advance, the present value when interest rates are 10% is
100 x **PVIFA**(3y, 10%) x 1.10
= 100 x 2.487 x 1.10 = **273.57**

While the tables can be used for most calculations, you should note that they are only provided for up to 60 periods -- for mortgage calculations, 300 period annuities are not uncommon (25 years of monthly payments) and so the formula (or a calculator) would be required.

$$PVIFA = \frac{1 - \left[\dfrac{1}{(1 + i)^n}\right]}{i} = \frac{1 - \left[\dfrac{1}{(1 + 0.10)^3}\right]}{0.10} = 2.487$$

Simple Conversion of Annuity In Advance from Ordinary Annuity Formula

Present Value of Annuity in Advance: $PVIFA^{ADV}$ (N years, i%)
= PVIFA(N years, i%) x (1 + i)

Future Value of Annuity in Advance: $FVIFA^{ADV}$(N years, i%)
= FVIFA(N years, i%) x (1 + i)

Use the above formulas to show that for a three year $100 annuity in advance, if interest rates are 10%, the present value will be **273.55** and the future value at the end of three years will be **364.09**, as found in the previous section.

I. Growing Annuities

While annuities are defined as a constant stream of cash flows for a defined number of years, another common cash flow stream is one that grows at a constant rate each year, e.g., the first cash flow is $100, and it grows at the rate of 8% each year for 2 years. If the appropriate interest rate for this cash flow stream is 10%, the present value of this stream can be calculated using the **PVIFGA** formula derived at the end of this lesson, where **PVIFGA** is the **p**resent **v**alue **i**nterest **f**actor for a **g**rowing **a**nnuity, given by

$$PVIFGA = \frac{\left[1 - \frac{(1+g)^N}{(1+i)^N}\right]}{i - g}$$

with ***g*** being the annual rate of growth in the annual cash flow.

The time line for the above mentioned growing annuity is indicated in the figure to the right.

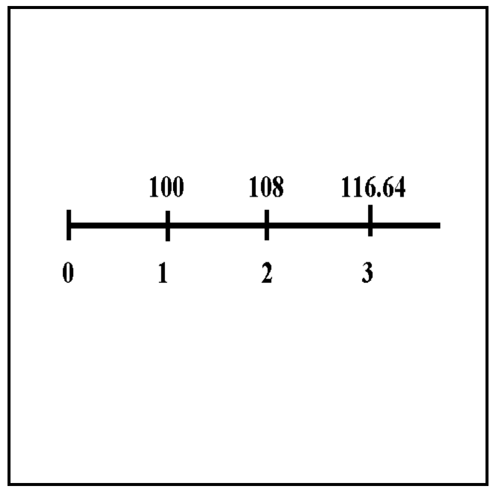


Figure 4 A Growing Annuity

You can work out the present value of this growing annuity either using the PVIFGA formula above, or by discounting each of the cash flows separately, i.e.,

$$\text{Present Value} = \frac{100}{(1+.10)^1} + \frac{108}{(1+.10)^2} + \frac{116.64}{(1+.10)^3}$$

$$= \underline{\underline{\mathbf{267.79}}}$$

Using the above formula,

Present Value = 100 x **PVIFGA**(3 y, g=8%, i=10%)
= 100 x 2.6779
= **<u>267.79</u>**

If the future value of such a stream was ever needed, you could simply employ the method used to determine the future value of an annuity due.

This formula for a growing annuity is not provided in the text, but you will find it helpful in working out some inflation related questions without having to resort to a spreadsheet.

J. Perpetuities

Each cash flow in a perpetuity is the same, and the cash flow stream goes on forever. A preferred share provides a nice example of such a stream: such a share will pay a dividend each year in the future, and since corporations have no finite life (theoretically at least), the stream of dividends could theoretically go on forever.

The formula for the present value of a perpetuity is derived at the end of this lesson:

$$\textbf{The present value of a Perpetuity} = \frac{\textbf{Cash Flow}}{\textbf{Interest rate}}$$

You might think that a stream that goes on forever, or even a very long time, should have a very large present value. Application of the above formula will indicate that this is not the case. Most of the value actually arises from the cash flows received over the first 25 to 30 years, as indicated in the exercise below.

Exercise:

Rusty Nails is evaluating the present value of $8 to be received every year for different time spans. If the appropriate discount rate is 8%, show that the proportion of an infinite stream would be received if the cash flow went on for each of the designated number of years:

Number of years	Proportion of Perpetual Stream Received
5	31.9%
10	53.7
20	78.5
25	85.3
30	90.1
50	97.9
100	99.95

{ **Hint**: *First calculate the present value of a perpetuity which pays $8 every year at the 8% discount rate; then calculate the present value of an annuity of $8 each year for, say 5 years, discounted at 8%*}

K. Growing Perpetuities

In a growing perpetuity, each cash flow grows at a constant rate, and the stream goes on forever.

The formula to calculate the present value of such a stream is

$$\mathbf{PV = \frac{CF}{K - G}}$$

where CF is the next year's cash flow
K is the appropriate discount rate
G is the constant growth rate at which each year's cash flow grows

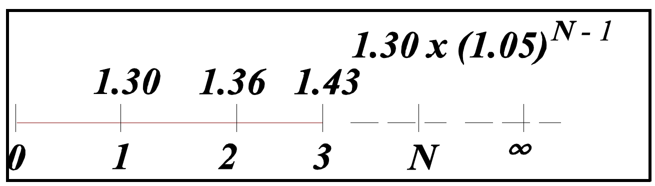
For this formula to work properly, the growth rate must be less than the discount rate.

For example,

Next year's cash flow is expected to be \$1.30 and cash flows are expected to grow at the rate of 5% per annum into the indefinite future. *What is the value of the stream if the discount rate is 10%?*

$$PV = \frac{\$1.30}{0.10 - 0.05} = \underline{\underline{\$26}}$$

The time line for such a stream is indicated in the diagram to the right.



L. Other Cases of Structured Cash Flows

In all of the previous examples dealing with annuities and perpetuities, the first cash flow was always either immediately received or received in one year. There are many common variations in which cash flows may have a regular pattern which allows us to use the annuity or perpetuity formulas, but in which the stream does not begin immediately.

i) Delayed Annuity or Delayed Growing Annuity

While this stream of cash flows contains an embedded annuity (or growing annuity), the annuity does not start until some later time.

Example:

John expects that his son, Jack, will require $10,000 at the start of each year during his 4 years in university. He expects his son to enter university in 4 years.

How much does he need to put away today to meet his son's needs if he can earn 8% p.a. on all investments?

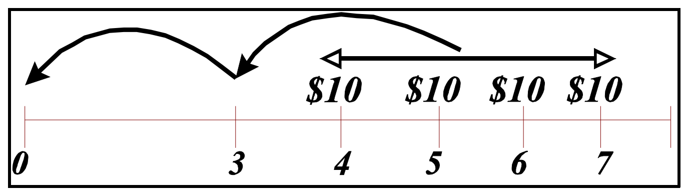


Figure 6 Deferred Annuity

The Present Value of a regular annuity collapses to the year before the first cash flow.

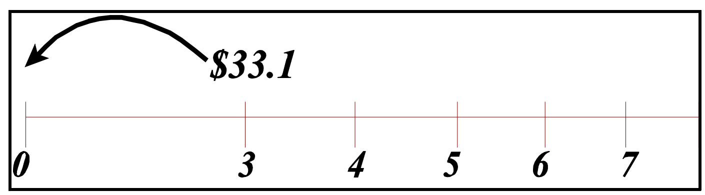


Figure 7 Present Value As of Year 3 of Tuition

So, the present value of the required tuition payments can be calculated as follows:

$10 x PVIFA(4 years, 8%) = **$33.1** { in thousands

Note that this is the present value as of year 3 (not time 0 when John is making the deposit), as indicated in Figure 12.

To determine the amount he needs to deposit today, we must take the present value of the $33.1 discounted by 3 years at 8%.

$33.1 x PVIF(3 years, 8%) = **$26.29** { in thousands

Alternatively, the calculation could all be done in one step as follows, as shown in Figure 11:

PV(0) = $10 x PVIFA(4 years, 8%) x PVIF(3years, 8%) = **$26.29** { in thousands

Example 2:

If John's daughter, Jill is expected to attend university in 6 years, and John expects to have to pay $12,000 per year for her, how much does it have to deposit today for both Jack and Jill if he can earn 8% on all investments?

PV(5) of Jill's expense stream = $12 x PVIFA(4 years, 8%) = **$39.7** { in thousands
PV(0) of Jill's expense stream = 39.7 x PVIF(5 years, 8%) = **$27.05** { in thousands

Total Investment today for Jack and Jill: $26.29 + $27.05 = **$53.34**

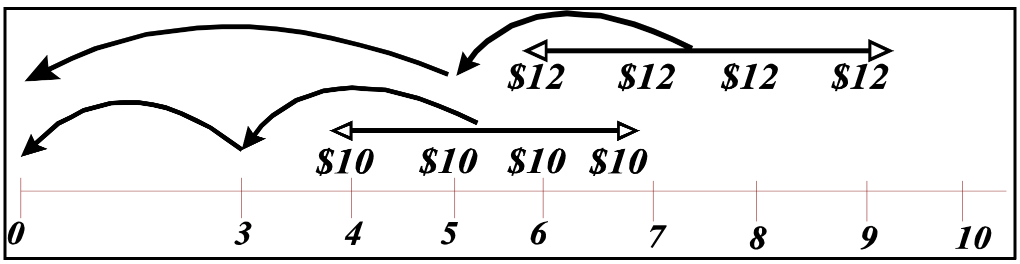


Figure 8 The PV of Two Deferred Annuities Arising Simultaneously

ii) Delayed Perpetuity or Delayed Growing Perpetuity

A stream of this sort should be dealt with in the same way as the deferred annuity stream, i.e., first, determine the present value of the perpetuity when it begins, and then discount that value to the present.

Example:

Turbo Enterprises has decided to discontinue its dividend to common shareholders for the next 3 years as a result of an extremely favourable investment opportunity that has arisen. It is expected that when the dividend does begin again at the end of year 3, it will be $2 and will be able to sustain that dividend payment forever. The company's discount rate is 10%. What should be the current value of the shares?

The value of the shares as of the start of year 3 will be $2/0.10 = $20
Today's value of those shares will be $20 x PVIF(3y, 10%) = **$15.03**

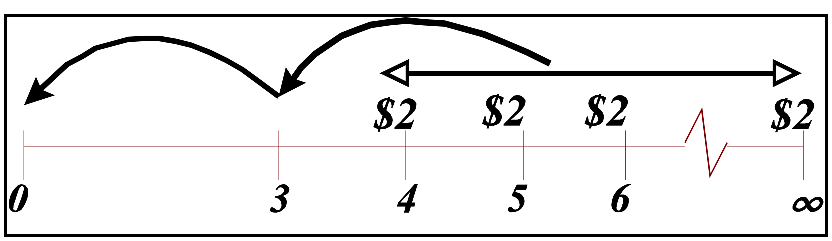


Figure 9 Present Value of a Deferred Perpetuity

iii) Infrequent Annuities

So far, in discussing annuities, we have assumed that the regular cash flows occurred every year. In an infrequent annuity, the cash flows are the same, but they occur less frequently than once per year.

Example:

You plan to purchase a new \$35,000 car every 5 years for the next 30 years starting in 5 years. *Calculate the PV of your future car purchases if your opportunity cost of capital is 7.0% per annum.*

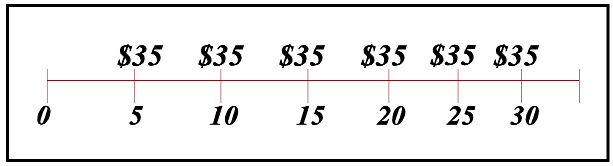


Figure 10 An Infrequent Annuity

The time line for this expected expenditure stream is shown in the above figure. Although it looks similar to a typical annuity in arrears, note that the cash outflows are not annual, but every 5 years.

To deal with this slightly different structure, we need to do the following,

1. **Determine the compound interest rate over the period between purchases**

   $R = (1 + R_0)^{N} - 1 = (1 + 0.07)^{5} - 1 =$ **40.25%**

   where R is the effective rate over the period between purchases

   R_0 is the annual interest rate (or opportunity cost of capital)

2. **Calculate the number of purchases**

   $T = T_0 / N$ = 30 years / 5 year = 6 purchases

   where T_0 is the total period of time over which the stream will occur

   N is the frequency of the cash flows over time

   and T is the total number of cash flows that will occur

3. **Calculate the present value of the stream (as though it were a regular annuity)**

   PV(0) = \$35 x PVIFA(6, 40.25%) = **\$75,531**

M. Extended Applications of Time Value To A Typical Personal Finance Problem

Example 1:

Norman has 10 years to go until retirement (at age 65) and he currently has an investment portfolio worth $500,000.

Everyone in Norman's family has lived long lives and he expects to survive to age 90. He wants to be able to withdraw $80,000 at the start of each year in retirement and to donate $100,000 to the *Red Cross* at his death.

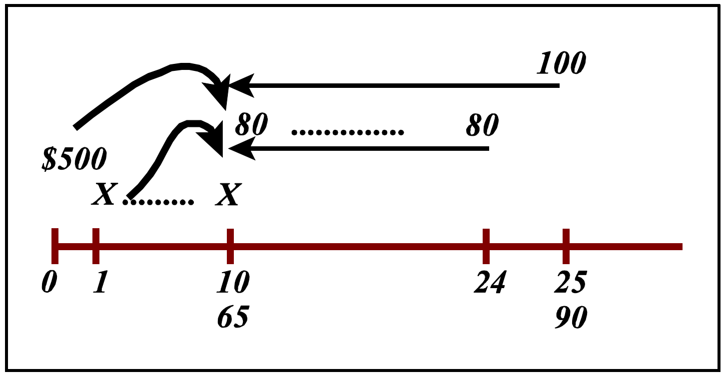


Figure 11 A Retirement Planning Problem

Norman expects to be able to earn 4.5% every year on all invested funds. *How much will he have to invest at the end of each year to meet his retirement goals?* (Ignore taxes).

This problem is a combination of present value and future value concepts, and is one of the more complicated time value problems that you might encounter in personal finance. If you understand this problem, others will seem quite trivial.

Norman's age 65 (year 10 in the exercise), his retirement time is likely the best place to centre all of the cash flows as this will automatically determine how much he will have to have at that time to meet his retirement goals. This is a ***present value problem*** over his retirement years.

PV(Withdrawals)	80,000 x **PVIFA**^{ADV}(25 years, 4.5%)	1239638
PV(Donation)	+ 100,000 x PV(26 years, 4.5%)	31840
	Total Needs at Retirement	**1271479**
PVIFA^{ADV} means Present Value Factor for an Annuity in Advance		

Norman must accumulate **$1,271,479** by age 65 to meet his retirement goals.

To do this, he will continue to invest and earn a rate of return on the existing portfolio as well as the incremental annual required investment. This is a ***future value problem*** over his pre-retirement (working) years.

500,000 x FVIF(10years, 4.5%) + X x FVIFA(10 years, 4.5%) = **$1,271,479**

where X is the ***additional annual investment*** required to meet the retirement goals

Solve for X: **X = $40,282**

Example 2:

Beverly and Brian were concerned with the rising cost of university education, so on the day that their son Billy was born, they deposited $2,500 into a savings account and pledged to make deposits every year on his birthday to help with his future post secondary education.

Deposits were indeed made on the first four birthdays, but no funds were available when Billy turned 5, and $5,000 had to be withdrawn when he turned 6 and 7. Thereafter, up until Billy turned 18, they were able to deposit $4,000 into an investment fund for him. All investments were able to earn 5% per annum.

How much was in the savings account the day following Billy's eighteenth birthday?

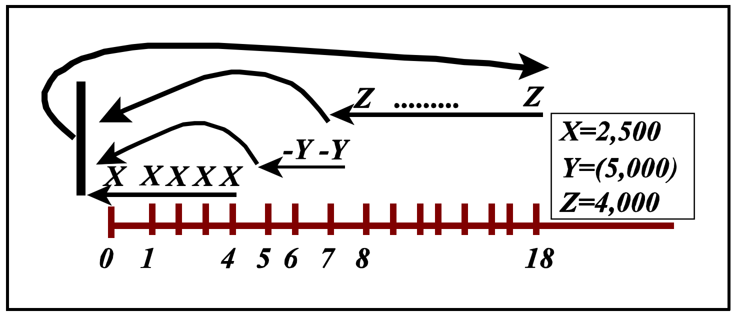


Figure 12 An Educational Savings Program

This is clearly a **future value problem** as the question asks how much will have accumulated by the time Billy turns 18.

However, since we are dealing with several annuities which last for only a limited period of time, I have chosen to determine what the present value of all of these amounts are as of time 0, and then determine the future value of that total amount as if his 18th birthday, as indicated in Figure 17.

Present Value of Savings on the first four birthdays and the date of birth as of Time 0

2,500 x PVIFA^{ADV}(5years, 5%) = 11,365

Present Value of Withdrawals of $5,000 on birthdays 6 and 7 as of Time 0

- 5,000 x PVIFA(2years, 5%) x PVIF(5years, 5%) = -7,284

Present Value of $4,000 savings form ages 8 to 18 as of Time 0

4,000 x PVIFA(11 years, 5%) x PVIF(7years, 5%) = 23,613

Present Value of All Cash Flows as of Time 0 **27,693**

The Future Value of this amount is

27,683 x FVIF(18 years, 5%) = **66,648**

This problem reinforces the interrelationship between present and future value as shown in Figure 1.

Appendix: Deriving Formulas for Annuities and Perpetuities

You are not responsible for the derivation of these formulas: they are provided here simply to indicate that they are derived from some simple arithmetic principles.

You may recall a problem that you faced in high school mathematics, about the sum of an infinite converging series, e.g.,

What is the sum of the following series:

$$2 + 1 + ½ + 1/4 + ...$$

And you may remember the answer to be 4.

An algebraic presentation of such a problem would be

$$a + a\mathbf{g} + a\mathbf{g}^2 + a\mathbf{g}^3 + ...$$

where **a** is the first term, and each subsequent term grows by the amount g

To obtain a result that is not infinite, the series must converge, which simply means that g<1.

The sum of such a series is given by the equation $S_\infty = \dfrac{a}{1 - g} \quad with \ g < 1$

and if you plug the values into the numerical series above, $S_\infty = 2/(1-1/2) = \underline{\underline{4}}$

What if you only wanted the sum of the first N terms of the series?

In that case the equation becomes

$$S_N = \frac{a - gL}{1 - g} \quad with \ g < 1 \ and \ L \ the \ last \ term \ of \ interest$$

Applying this formula to calculate the sum of the first 3 terms in the numerical equation above gives, S_N = (2- ½ x ½) /(1- ½) = 3.50 which is the same as adding 2 + 1+ 0.50, the individual series elements. Of course, the formula is more efficient if you wanted to get the sum of the first 20 or 30 terms.

Now to apply this idea to a series of discounted cash flows over time:

What is the present value of an annuity of $1 per year, each year for N years?

We are trying to get the sum of the following discounted cash flow stream:

$$PV = \frac{1}{(1+K)^1} + \frac{1}{(1+K)^2} + \;....\; + \frac{1}{(1+K)^N} \quad \textit{where K is the discount rate}$$

If you apply the S_N formula developed above, recognizing that the growth factor between terms is simply 1/(1+K), then the present value for an annuity becomes

$$PVIFA = \frac{\left[1 - \dfrac{1}{(1+K)^N}\right]}{K}$$

just like the formula presented in the table in Appendix 1-B (p.40) for the present value of an annuity {***Present Value of a Series of Equal Amounts* (An Annuity)**} .

The same arithmetic can be applied to determine the present value of a growing annuity. Assume that each year the $1 grows at the rate F. The growing annuity stream becomes:

$$PV = \frac{1}{(1+K)^1} + \frac{1 \ x \ (1+F)^1}{(1+K)^2} + \;....\; + \frac{1 \ x \ (1+F)^{N-1}}{(1+K)^N}$$

and the equation to determine the present value of this growing annuity is

$$PVIFGA = \frac{\left[1 - \dfrac{(1+F)^N}{(1+K)^N}\right]}{K - F}$$

If the cash flow stream goes on forever, i.e., a perpetuity, the S_∞ formula must be invoked

$$S_\infty = \frac{a}{1 - g} \quad with \ g < 1$$

For the cash flow stream of $1 each year forever,

$$PV = \frac{1}{(1+K)^1} + \frac{1}{(1+K)^2} + \;....\; + \frac{1}{(1+K)^N} + \;.....\; +$$

The present value is given by

$$S_\infty = \frac{a}{1 - g} = \frac{\dfrac{1}{1 + K}}{1 - \dfrac{1}{(1 + K)}} = \frac{1}{K}$$