Midterm MATH 251, October 22, 2015

Justify all answers

Problem 1 [5 pt] Let V be the vector space indicated below, case by case. In each case find if the specified vector $\mathbf{v}$ belongs to the span of the given set S.

$$V = \mathbb{R}^4; \quad \mathbf{v} = (2,1,1,1); \quad S := \{(1,0,1,1),(1,1,1,1),(2,1,2,2),(0,1,0,0),(0,0,0,3)\} \tag{1}$$

$$V = \mathcal{P}_4(\mathbb{R}) \ , \quad \mathbf{v} = 3x^3+2x-1, \quad S := \left\{x^4+2, 2x-1, x^3, x^3+3\right\} \tag{2}$$

Problem 2 [5pt] Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three *linearly independent* vectors in the vector space V. Show that the three new vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ given below are also linearly independent

$$\mathbf{x} = 2\mathbf{u}+\mathbf{v}+\mathbf{w} \ , \quad \mathbf{y} = \mathbf{u}-\mathbf{v} \ , \quad \mathbf{z} = \mathbf{u}+\mathbf{w}. \tag{3}$$

Problem 3 [5 pt] Let $V = Mat_{2\times 3}(\mathbb{R})$ and consider the following subspaces:

$$W_1 := \left\{ \left[\begin{array}{ccc} a & a-b & a+b \\ c+d & d & d \end{array}\right] \ a,b,c,d \in \mathbb{R} \right\} \ , \qquad W_2 := \left\{ \left[\begin{array}{ccc} f & -f & g \\ e & e & \ell - e \end{array}\right] \ e,f,g,\ell \in \mathbb{R} \right\} \tag{4}$$

Find bases and dimensions of $W_1, \ W_2, \ W_1+W_2, \ W_1 \cap W_2$.

Problem 4 [5 pt] Let $T : V \to W$ be a linear transformation between two finite dimensional vector spaces. Suppose that $\dim V = (\dim W) - 2$ and that $\operatorname{rank}(T) = (\dim W) - 3$. Can the map be one-to-one? If yes give an example, if not explain why.

Problem 5 [5 pt] Let $T : \mathcal{P}_2 \to \mathbb{R}^2$ be given by $T(p(x)) = \langle p(3), \int_0^1 p(x)\, \mathrm{d}x \rangle$. Let $\beta = \{x+2, x-3, x^2\}$ be a basis of $\mathcal{P}_2$ and $\alpha = \{1, x, x^2\}$ another basis. Let $\gamma = (\langle 1,1\rangle, \langle 1,-1\rangle)$ be a basis of $\mathbb{R}^2$. Compute $[T]_\beta^\gamma, \quad [T]_\alpha^\gamma$.

Problem 6 [5 pt]

Consider the transformation $T : \mathcal{P}_2 \to \mathcal{P}_4$ where $\mathcal{P}_n$ denotes the finite dimensional vector space consisting of polynomials of degree up to n:

$$T(p(x)) = p(x^2) + 3p(x-2) \tag{5}$$

Note: here $p(x-2)$ means the shift of variable, for example if $p(x) = x^2+2$ then $p(x-2) = (x-2)^2+2 = x^2-4x+6$; similarly $p(x^2)$ means the change of variable, for example if $p(x) = 2x^2+x$ then $p(x^2) = 2(x^2)^2+(x^2) = 2x^4+x^2$.

1. Show that T is linear;
2. Find $[T]_\beta^\gamma$ where $\beta = (1, x, x^2), \gamma = (1, x, x^2, x^3, x^4)$ are the standard ordered bases of $\mathcal{P}_2, \mathcal{P}_4$.

Problem 7 [Bonus 3 pt] Let $T : V \to W$ and $U : V \to W$ be two linear transformations between the indicated vector spaces. Prove that $\mathbf{R}(U+T) \subseteq \mathbf{R}(U) + \mathbf{R}(T)$, where $\mathbf{R}(U), \ \mathbf{R}(T)$ denote the *ranges* of the indicated transformation. Give an example where the inclusion is strict.

Solution to Problem 1 **First case**. We try to write them as a linear combination (in column form for brevity)

$$\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b+2c \\ b+c+d \\ a+b+2c \\ a+b+2c+3e \end{pmatrix} \tag{6}$$

The system clearly has no solution because the two equtions marked in red are incompatible. Thus the vector v is not in the span.

Second case

$$3x^3+2x-1 = a(x^4+2)+b(2x-1)+cx^3+d(x^3+3) = ax^4+(c+d)x^3+2bx+2a-b+3d \tag{7}$$

Equating the coefficients of both sides we get the system

$$\left\{\begin{array}{c} a=0 \\ c+d=3 \\ 2b=2 \\ 2a-b+3d=-1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} a=0 \\ c=3 \\ b=1 \\ d=0 \end{array}\right. \tag{8}$$

Thus

$$\mathbf{v} = w_2 + 3w_3 \tag{9}$$

where w_1, w_2, w_3, w_4 are the four polynomials listed in S. $\square$

Solution to Problem 2 We try to write a linear combination of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ giving zero

$$a\mathbf{x}+b\mathbf{y}+x\mathbf{z} = a\left(2\mathbf{u}+\mathbf{v}+\mathbf{w}\right)+b(\mathbf{u}-\mathbf{v})+c(\mathbf{u}+\mathbf{w}) = (2a+b+c)\mathbf{u}+(a-b)\mathbf{v}+(a+c)\mathbf{w} = \mathbf{0}. \tag{10}$$

Since the vectors u, v, w are independent, the above equation has solution iff

$$\left\{\begin{array}{c} 2a+b+c=0 \\ a-b=0 \\ a+c=0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} a=0 \\ b=0 \\ c=0 \end{array}\right. \tag{11}$$

Thus they are linearly independent. $\square$

Solution to Problem 3

Basis for W_1 Any matrix in W_1 is

$$M = \begin{bmatrix} a & a-b & a+b \\ c+d & d & d \end{bmatrix} = a\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tag{12}$$

The four matrices appearing in the linear combination above are linearly independent because the l.h.s. is zero iff $a=b=c=d=0$. Thus they form a basis and $\dim W_1 = 4$.

Basis for W_2. Any matrix in W_2 is

$$M = \begin{bmatrix} f & -f & g \\ e & e & \ell-e \end{bmatrix} = f\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + g\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \ell\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} \tag{13}$$

The four matrices appearing in the linear combination above are linearly independent because the l.h.s. is zero iff $f=g=\ell=e=0$. Thus they form a basis and $\dim W_2 = 4$.

Sum space $W_1 + W_2$. A matrix in the sum $W_1 + W_2$ has the form

$$\begin{bmatrix} a+f & a-b-f & a+b+g \\ c+d+e & d+e & d+\ell-e \end{bmatrix} \tag{14}$$

We claim that any matrix in $V = Mat_{2\times 3}$ can be expressed in the above form. To see it let $M = \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix}$. Equating the entries we have

$$\begin{bmatrix} a+f & a-b-f & a+b+g \\ c+d+e & d+e & d+\ell-e \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \Rightarrow \begin{cases} a+f=A \\ a-b-f=B \\ a+b+g=C \\ c+d+e=D \\ d+e=E \\ \ell+d-e=F \end{cases} \tag{15}$$

We can eye a solution by setting $f = 0 = e$ (for example) and then

$$\begin{cases} a=A \\ b=A-B \\ g=C-2A+B \\ c=D-E \\ d=E \\ \ell=F-E \end{cases} \tag{16}$$

Thus the $\dim(W_1 + W_2) = \dim V = 6$ and a basis is for example the standard basis.

Intersection Equating the defining equations we have

$$\begin{bmatrix} a & a-b & a+b \\ c+d & d & d \end{bmatrix} = \begin{bmatrix} f & -f & g \\ e & e & \ell-e \end{bmatrix} \tag{17}$$

which yields the system

$$\begin{cases} a=f \\ a-b=-f \\ a+b=g \\ c+d=e \\ d=e \\ d=\ell-e \end{cases} \Rightarrow \begin{cases} a=f \\ b=2f \\ g=3f \\ c=0 \\ d=e \\ \ell=2e \end{cases} \Rightarrow M = \begin{bmatrix} f & -f & 3f \\ e & e & e \end{bmatrix} = f\begin{bmatrix} 1 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tag{18}$$

The two matrices on the right side above are linearly independent because $M = 0$ iff $f = e = 0$. Thus $\dim(W_1 \cap W_2) = 2$ and the basis consists of the mentioned two matrices. $\square$

Solution to Problem 4 From the nullity+rank theorem

$$nul(T) + rk(T) = \dim V \tag{19}$$

we have

$$nul(T) = \dim V - rk(T) \tag{20}$$

Since $rk(T) = \dim W - 3$ and $\dim V = \dim W - 2$ we have

$$nul(T) = \dim W - 2 - \dim W + 3 = 1 \tag{21}$$

The map cannot be one-to-one because the kernel (null-space) is nontrivial. □

Solution to Problem 5

$$T(x+2) = \big(3+2, \int_0^1 (x+2)\,\mathrm{d}x\big) = \big(5, \frac{5}{2}\big) = \frac{15}{4}(1,1) + \frac{5}{4}(1,-1) \tag{22}$$

$$T(x-3) = \big(3-3, \int_0^1 (x-3)\,\mathrm{d}x\big) = \big(0, -\frac{5}{2}\big) = -\frac{5}{4}(1,1) + \frac{5}{4}(1,-1) \tag{23}$$

$$T(x^2) = \big(3^2, \int_0^1 x^2\,\mathrm{d}x\big) = \big(9, \frac{1}{3}\big) = \frac{28}{6}(1,1) + \frac{26}{6}(1,-1) \tag{24}$$

$$[T]_\beta^\gamma = \begin{bmatrix} \frac{15}{4} & -\frac{5}{4} & \frac{28}{6} \\ \frac{5}{4} & \frac{5}{4} & \frac{26}{6} \end{bmatrix} \tag{25}$$

$$T(x^0) = \big(3^0, \int_0^1 1\,\mathrm{d}x\big) = \big(1, 1\big) = 1(1,1) + 0(1,-1) \tag{26}$$

$$T(x) = \big(3, \int_0^1 x\,\mathrm{d}x\big) = \big(3, \frac{1}{2}\big) = \frac{7}{4}(1,1) + \frac{5}{4}(1,-1) \tag{27}$$

$$T(x^2) = \big(3^2, \int_0^1 x^2\,\mathrm{d}x\big) = \big(9, \frac{1}{3}\big) = \frac{28}{6}(1,1) + \frac{26}{6}(1,-1) \tag{28}$$

$$[T]_\alpha^\gamma = \begin{bmatrix} 1 & \frac{7}{4} & \frac{28}{6} \\ 0 & \frac{5}{4} & \frac{26}{6} \end{bmatrix} \tag{29}$$

□

Solution to Problem 6 The map is linear;

$$T((p+q)(x)) = (p+q)(x^2) + 3(p+q)(x-2) = p(x^2) + 3p(x-2) + q(x^2) + 3q(x-2) = T(p(x)) + T(q(x)) \tag{30}$$

$$T(\lambda p(x)) = \lambda p(x^2) + \lambda 3p(x-2) = \lambda(p(x^2) + 3p(x-2)) = \lambda T(p(x)) \tag{31}$$

Then:

$$\begin{aligned} T(1) &= 1+3 &&= 4; \qquad (32)\\ T(x) &= x^2 + 3(x-2) &&= -6 + 3x + x^2; \qquad (33)\\ T(x^2) &= x^4 + 3(x-2)^2 &&= 12 - 12x + 3x^2 + x^4 \qquad (34)\\ & && \qquad (35) \end{aligned}$$

Thus

$$[T]_\beta = \begin{bmatrix} 4 & -6 & 12 \\ 0 & 3 & -12 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{36}$$

□

Solution to Problem 7 If $\underline{w} \in \mathbf{R}(U+T)$ is an arbitrary vector in the indicated range, then there must exist $\underline{v} \in V$ such that $\underline{w} = (U+T)(\underline{v})$. Then

$$\underline{w} = (U+T)(\underline{v}) = U(\underline{v}) + T(\underline{v}) = \underline{w}_1 + \underline{w}_2 \tag{37}$$

where, by definition of range, $\underline{w}_1 \in \mathbf{R}(U)$, $\underline{w}_2 \in \mathbf{R}(T)$. Then, by definition of sum of vector subspaces $\underline{w} \in \mathbf{R}(U) + \mathbf{R}(T)$. Therefore we have the stated inclusion.

To give an example where the inclusion is strict consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ to be the identity and $U = -T$. Then $U+T$ is the zero map, and its range is trivial. On the other hand the ranges of U, T are the whole $\mathbb{R}^2$, and hence their sum is also $\mathbb{R}^2$. $\square$