MAT 2384 A Assignment #3 : Solutions

1. $y''' - 6y'' + 11y' - 6y = 0, \quad y(0)=6, \; y'(0)=14, \; y''(0)=36$

the characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$
by inspection, $\lambda = 1$ is a root, so $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda-1)(\lambda^2 - 5\lambda + 6)$
so $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$ $\qquad = (\lambda-1)(\lambda-2)(\lambda-3)$

the general solution is $y(x) = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

$y(0) = 6 \Rightarrow 6 = c_1 e^0 + c_2 e^0 + c_3 e^0 \Rightarrow c_1 + c_2 + c_3 = 6$ ①
$y'(x) = c_1 e^{x} + 2c_2 e^{2x} + 3c_3 e^{3x}$
$y'(0) = 14 \Rightarrow 14 = c_1 e^0 + 2c_2 e^0 + 3c_3 e^0 \Rightarrow c_1 + 2c_2 + 3c_3 = 14$ ②
$y''(x) = c_1 e^{x} + 4c_2 e^{2x} + 9c_3 e^{3x}$
$y''(0) = 36 \Rightarrow 36 = c_1 e^0 + 4c_2 e^0 + 9c_3 e^0 \Rightarrow c_1 + 4c_2 + 9c_3 = 36$ ③

③ − ① $\quad 3c_2 + 8c_3 = 30$ ④
② − ① $\quad c_2 + 2c_3 = 8$ ⑤

④ − 3×⑤ $\Rightarrow 2c_3 = 6 \Rightarrow c_3 = 3$
then $c_2 = 2$ and $c_1 = 1$

∴ the unique solution is $\boxed{y(x) = e^{x} + 2e^{2x} + 3e^{3x}}$

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2. $y^{(4)} + 5y'' + 4y = 0, \quad y(0) = -1, \; y'(0) = 2, \; y''(0) = 4, \; y'''(0) = -2$

the char. eq. is $\lambda^4 + 5\lambda^2 + 4 = (\lambda^2+1)(\lambda^2+4) = 0$
so $\lambda_{1,2} = \pm i$ and $\lambda_{3,4} = \pm 2i$
and the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x + c_3 \cos(2x) + c_4 \sin(2x)$$

$y(0) = -1 \Rightarrow -1 = c_1 \cos(0) + c_2 \sin(0) + c_3 \cos(0) + c_4 \sin(0)$
$\Rightarrow c_1 + c_3 = -1$ ①

$y'(x) = -c_1 \sin x + c_2 \cos x - 2c_3 \sin(2x) + 2c_4 \cos(2x)$

$y'(0) = 2 \Rightarrow 2 = -c_1 \sin(0) + c_2 \cos(0) - 2c_3 \sin(0) + 2c_4 \cos(0)$

$\Rightarrow c_2 + 2c_4 = 2$ ②

$y''(x) = -c_1 \cos x - c_2 \sin x - 4c_3 \cos(2x) - 4c_4 \sin(2x)$

$y''(0) = 4 \Rightarrow 4 = -c_1 \cos(0) - c_2 \sin(0) - 4c_3 \cos(0) - 4c_4 \sin(0)$

$\Rightarrow -c_1 - 4c_3 = 4$ ③

$y'''(x) = c_1 \sin x - c_2 \cos x + 8c_3 \sin(2x) - 8c_4 \cos(2x)$

$y'''(0) = -2 \Rightarrow -2 = c_1 \sin(0) - c_2 \cos(0) + 8c_3 \sin(0) - 8c_4 \cos(0)$

$\Rightarrow -c_2 - 8c_4 = -2$ ④

① + ③ $\Rightarrow -3c_3 = 3 \Rightarrow c_3 = -1 \Rightarrow c_1 = 0$

② + ④ $\Rightarrow -6c_4 = 0 \Rightarrow c_4 = 0 \Rightarrow c_2 = 2$

∴ the unique solution is $\boxed{y(x) = 2\sin x - \cos(2x)}$

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3. $x^3 y''' - x^2 y'' + 2xy' - 2y = 0, \quad x > 0, \quad y(1) = -3, \ y'(1) = -7, \ y''(1) = -9$

the characteristic equation is $m(m-1)(m-2) - m(m-1) + 2m - 2 = 0$

or $m(m-1)(m-2) - m(m-1) + 2(m-1) = 0$

or $(m-1)(m(m-2) - (m-2)) = 0$

so $(m-1)^2(m-2) = 0 \Rightarrow m_{1,2} = 1, \ m_3 = 2$

so the general solution is $y(x) = c_1 x + c_2 x \ln x + c_3 x^2$

$y(1) = -3 \Rightarrow -3 = c_1(1) + c_2(1)\ln(1) + c_3(1)^2 \Rightarrow c_1 + c_3 = -3$ ①

$y'(x) = c_1 + c_2 \ln x + c_2 + 2c_3 x$

$y'(1) = -7 \Rightarrow -7 = c_1 + c_2 \ln(1) + c_2 + 2c_3(1) \Rightarrow c_1 + c_2 + 2c_3 = -7$ ②

$y''(x) = c_2/x + 2c_3$

$y''(1) = -9 \Rightarrow -9 = c_2/(1) + 2c_3 \Rightarrow c_2 + 2c_3 = -9$ ③

③ − ③ $\Rightarrow$ $c_1 = 2$ $\Rightarrow$ $c_3 = -5$ $\Rightarrow$ $c_2 = 1$

$\therefore$ the unique solution is $\boxed{y(x) = 2x + x\ln x - 5x^2}$

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4. $y'' - 4y = 3e^{2x} + 12x - 2$,

the corresponding homog. DE is $y''-4y=0$, which has characteristic equation $\lambda^2 - 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2$ and so $y_h(x) = c_1 e^{2x} + c_2 e^{-2x}$

$r(x) = 3e^{2x} + 12x - 2 \Rightarrow y_p(x) = axe^{2x} + bx + c$ (Mod Rule used)
then $y_p'(x) = ae^{2x} + 2axe^{2x} + b$
and $y_p''(x) = 4ae^{2x} + 4axe^{2x}$

so $y_p'' - 4y_p = 4ae^{2x} + 4axe^{2x} - 4(axe^{2x} + bx + c)$
$= 4ae^{2x} - 4bx - 4c = r(x) = 3e^{2x} + 12x - 2$
so $a = 3/4$, $b = -3$ and $c = 1/2$
thus $y_p(x) = \frac{3}{4}xe^{2x} - 3x + \frac{1}{2}$

then the general solution is $\boxed{y_g(x) = c_1 e^{2x} + c_2 e^{-2x} + \frac{3}{4}xe^{2x} - 3x + \frac{1}{2}}$

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2. $y'' - 5y' + 6y = 2e^x + 40\cos x$,

corresp. homog DE is $y'' - 5y' + 6y = 0$, which has char. eq.
$\lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3$
and so $y_h(x) = c_1 e^{2x} + c_2 e^{3x}$

$r(x) = 2e^x + 40\cos x \Rightarrow y_p(x) = ae^x + b\cos x + c\sin x$

then $y_p'(x) = ae^x - b\sin x + c\cos x$ and $y_p''(x) = ae^x - b\cos x - c\sin x$
so $y_p'' - 5y_p' + 6y_p = ae^x - b\cos x - c\sin x - 5(ae^x - b\sin x + c\cos x)$
$+ 6(ae^x + b\cos x + c\sin x)$
$= 2ae^x + (5b - 5c)\cos x + (5c + 5b)\sin x$
$= r(x) = 2e^x + 40\cos x$

so we have $a=1$, $\left.\begin{array}{l}5b-5c=40\\ 5b+5c=0\end{array}\right\}$ $b=4,\ c=-4$

and so $y_p(x) = e^x + 4\cos x - 4\sin x$

the general solution is $\boxed{y_g(x) = c_1 e^{2x} + c_2 e^{3x} + e^x + 4\cos x - 4\sin x}$

6. natural cubic spline to $(-1,1)$, $(0,0)$, $(1,2)$
so the cubics are $S_0(x) = a_0 + b_0 x + c_0 x^2 + d_0 x^3$ on $[-1,0]$
and $S_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3$ on $[0,1]$

$S_0(-1) = 1 \Rightarrow a_0 - b_0 + c_0 - d_0 = 1$
$S_0(0) = 0 \Rightarrow a_0 = 0$
$S_1(0) = 0 \Rightarrow a_1 = 0$
$S_1(1) = 2 \Rightarrow a_1 + b_1 + c_1 + d_1 = 2$

$S_0'(0) = S_1'(0) \Rightarrow b_0 = b_1$

$S_0''(0) = S_1''(0) \Rightarrow c_0 = c_1$

$S_0''(-1) = 0 \Rightarrow 2c_0 - 6d_0 = 0 \Rightarrow c_0 - 3d_0 = 0$
$S_1''(1) = 0 \Rightarrow 2c_1 + 6d_1 = 0 \Rightarrow c_1 + 3d_1 = 0$
which means $d_1 = -d_0 = -1/3\, c_0$

the solution is $a_0 = a_1 = 0$, $b_0 = b_1 = 1/2$,
$c_0 = c_1 = 9/4$, $d_0 = 3/4$, $d_1 = -3/4$

or
$$\boxed{\begin{array}{l} S_0(x) = \frac{1}{2}x + \frac{9}{4}x^2 + \frac{3}{4}x^3 \\ \\ S_1(x) = \frac{1}{2}x + \frac{9}{4}x^2 - \frac{3}{4}x^3 \end{array}}$$