& DGD2 (TBT-0019, 2:30 pm)

MAT1341B–DGD 1 (~~FTX 137, 2:30pm~~) Test 2, 2016

October 17, 2016. Duration: 75 minutes.

Instructor – Saeid Molladavoudi

Key

Family Name:______________________

First Name:______________________

Student number:______________________

Multiple choice answers → (1, 2)

For the marker's use only → (3, 4, 5)

1	E
2	C
subtotal	
3	
4	
5	
[Bonus] 6	
Total	

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. You have 75 minutes for this test.
2. Read each question carefully, and **answer all questions in the space provided after each question.** For questions 4 to 6, you may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
3. This is a closed book exam, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted. All implanted cyber devices not necessary for life-support must be disabled at the beginning of the exam.
4. Questions 1 and 2 are multiple choice. These questions are worth 1 point each and no part marks will be given. Please record your answers in the space provided above.
5. Questions 3 – 5 and are worth 6 points each, and part marks can be earned. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.** Question 6 is a bonus question and is worth 3 points. It is much more difficult to earn bonus points, so complete and check the rest of the paper before trying this question.
6. Where it is possible to check your work, do so.
7. Good luck! Bonne chance!

1. Which of the following are subspaces of $\mathbf{R}^3$?

$U = \{(x-y,\ x+y,\ x-y) \mid x,\ y \in \mathbf{R}\}$ $= \text{span}\{(1,1,1), (-1,1,-1)\}$ ✓
$V = \{(x,\ y,\ -y) \mid x,\ y \in \mathbf{R}\}$ $= \text{span}\{(1,0,0), (0,1,-1)\}$ ✓
$W = \{(x^2,\ y,\ x+y) \mid x,\ y \in \mathbf{R}\}$ ✗
$X = \{(x,y,z) \mid x-y=0\}$ $= \text{span}\{(-1,1,0), (0,0,1)\}$ ✓

A. U and V only
B. U and W only
C. W and X only
D. U, W and X only
E. U, V and X only
F. V and W only

2. Which of the following statements is/are true?

✗ I. The span of any two different vectors in $\mathbf{R}^2$ is all of $\mathbf{R}^2$. vectors could be parallel.

✓ II. The set $\{(1,2)\}$ spans a line through the origin in $\mathbf{R}^2$.

✓ III. In a vector space V which contains u, v and w, if every vector in V is a linear combination of v and $u+v+w$, then $V = \text{span}\{u,v,w\}$. (below)

✗ IV. The set $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ spans $\mathbf{M}_{2\times 2}(\mathbf{R})$. $\dim(M_{2\times2}(\mathbb{R})) = 4$

✗ V. The set $\{(1,0,1), (0,2,3)\}$ spans $\mathbf{R}^3$. $\dim(\mathbb{R}^3) = 3$

$\dim(V) \leq$ # vectors in a spanning set

A. I only
B. II only
C. II and III only
D. IV and V only
E. I, II, IV, and V only
F. All of the statements are true.

$\vec{x} \in V$, $\vec{x} = a\vec{v} + b(\vec{u}+\vec{v}+\vec{w}) = a\vec{v} + b\vec{u} + b\vec{v} + b\vec{w}$
$= (a+b)\vec{v} + b\vec{u} + b\vec{w}$ and so the set

$\{\vec{u}, \vec{v}, \vec{w}\}$ spans V.

3. Let $W = \{(x, y, z) \in \mathbf{R}^3 \mid x + y - z = 0\}$.

a) Explain *very briefly* why W is a subspace of $\mathbf{R}^3$. (*You will not need to use the Subspace Test.*)

b) Find a spanning set for W.

c) Find a basis for W.

d) Give a complete geometric description of W.

(Remember that you must justify your answers.)

a) W is a plane through $\vec{0}$ and so a subspace of $\mathbb{R}^3$.

b) $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0\}$

$= \{(x, y, x+y) \mid x, y \in \mathbb{R}\}$

$= \{(x, 0, x) + (0, y, y) \mid x, y \in \mathbb{R}\}$

$= \{x(1,0,1) + y(0,1,1) \mid x, y \in \mathbb{R}\}$

$= \text{Span}\{(1,0,1), (0,1,1)\}$

c) $(1,0,1)$ & $(0,1,1)$ are L.I., since they are not multiple of each other. So, they form a basis for W. $\{(1,0,1), (0,1,1)\}$ is a basis.

d) W is a plane going through the origin with the normal vector $(1, 1, -1)$.

4. Let $\mathbf{M}_{2\times 2}(\mathbf{R})$ denote the vector space of 2 by 2 matrices with real entries, and define

$$U = \left\{ \begin{bmatrix} a & b \\ -b & c \end{bmatrix} \in \mathbf{M}_{2\times 2}(\mathbf{R}) \mid a, b, c \in \mathbf{R} \right\}.$$

a) Either check that U is closed under addition, or express U in another form so you can simply state a theorem that guarantees that U is a subspace.

(For parts (b) and (c) you may assume that U is a subspace of $\mathbf{M}_{2\times 2}(\mathbf{R})$.)

b) Find a basis for U, and hence find $\dim U$.

c) Give a basis for U, different from the one you gave in (b).

(Remember that you must justify your answers.)

a) $U = \left\{ \begin{pmatrix} a & b \\ -b & c \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \mid a, b, c \in \mathbb{R} \right\}$

$= \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$

$= \cancel{\text{span}} \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$

$= \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

All spans are subspaces, so is U.

b) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ spans U.

$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a & b \\ -b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{a = 0,\ b = 0,\ c = 0}$

They are LI too. So, they form a basis for U.
$\dim(U) = 3$.

c) Claim $\left\{ 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

is also a basis.

<u>Proof:</u> Let's $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

and an arbitrary matrix $D \in U$:

Since $D \in U$, $D = aA + bB + cC$

$$= \left(\frac{a}{2}\right)(2A) + \left(\frac{b}{2}\right)(2B) + \left(\frac{c}{2}\right)(2C)$$

$$= a'(2A) + b'(2B) + c'(2C)$$

$a', b', c' \in \mathbb{R}$

$\Rightarrow U = \{ a'(2A) + b'(2B) + c'(2C) \mid a', b', c' \in \mathbb{R} \}$

$= \text{Span}\{2A, 2B, 2C\}$; They span U

Are they LI?

$$a'(2A) + b'(2B) + c'(2C) = \underset{\in M_{2\times2}(\mathbb{R})}{0}$$

$$\to \underbrace{(2a')}_{a''}A + \underbrace{(2b')}_{b''}B + \underbrace{(2c')}_{c''}C = 0$$

$$\to a''A + b''B + c''C = 0, \quad a'', b'', c'' \in \mathbb{R}$$

but $\{A, B, C\}$ are LI, so <u>$a'' = b'' = c'' = 0$</u>

is the only solution to the dependence relation, so $\{2A, 2B, 2C\}$ are LI and so another basis for U.

5. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you <u>must give an explicit example - with numbers, or functions, as is appropriate!</u>

- If you say the statement is always true, you must give a clear explanation.

a) $X = \{f \in \mathbf{F}(\mathbf{R}) \mid f(x) \leq 0 \quad \text{for all } x \in \mathbf{R}\}$ is a subspace of $\mathbf{F}(\mathbf{R})$

not closed under scalar multiplication

let's $a \in \mathbb{R}$, such that $a < 0$

$\Rightarrow \underbrace{af(x)}_{g(x)} \geqslant 0 \Rightarrow g(x) \notin X.$

ANSWER $\boxed{\text{False}}$

b) If V is a vector space and $\{v_1, v_2\}$ spans V, then $\{v_1, v_2, v_3\}$ spans V for *any* vector $v_3 \in V$.

$V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$, $\vec{v}_3 \in V \Rightarrow \vec{v}_3 = a\vec{v}_1 + b\vec{v}_2$

$\Rightarrow V = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ ✓

ANSWER $\boxed{\text{True}}$

5 (cont.).

$-a = c$

c) $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{2\times 2}(\mathbf{R}) \mid a + c = 0 \right\}$ is a subspace of $\mathbf{M}_{2\times 2}(\mathbf{R})$.

$U = \left\{ \begin{pmatrix} a & b \\ -a & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \mid a, b, d \in \mathbb{R} \right\}$

$U = \left\{ a\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid a, b, d \in \mathbb{R} \right\}$

$U = \operatorname{span}\left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

All spans are subspaces.

ANSWER $\boxed{\text{True.}}$

d) If u_1, u_2 and u_3 are vectors in a vector space U, and $\{u_1, u_2, u_3\}$ is linearly independent, then $\dim U = 3$.

number of elements in any LI set in $U \leq \dim(U)$

ANSWER $\boxed{\text{False.}}$

6. [Bonus] Suppose that u, v, w are non-zero vectors in $\mathbf{R}^{2016}$ such that $u \cdot v = u \cdot w = v \cdot w = 0$. Prove that $\{u, v, w\}$ is linearly independent.

(Your proof must work for *all* choices of non-zero vectors in $\mathbf{R}^{2016}$ such that $u \cdot v = u \cdot w = v \cdot w = 0$ — do not choose them yourself. Use the definition. No 'geometric' argument - e.g. "they are not co-planar" - will suffice, and in any case is meaningless to low-dimensional beings like your instructor and marker.)

Let's write down the dependence relation:

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$$

Then, take dot products with $\vec{u}$, then $\vec{v}$ & then $\vec{w}$, we obtain:

$$a\,\vec{u}\cdot\vec{u} + b\,\vec{v}\cdot\vec{u} + c\,\vec{w}\cdot\vec{u} = \vec{0}$$

$$a\,\vec{u}\cdot\vec{v} + b\,\vec{v}\cdot\vec{v} + c\,\vec{w}\cdot\vec{v} = \vec{0}$$

$$a\,\vec{u}\cdot\vec{w} + b\,\vec{v}\cdot\vec{w} + c\,\vec{w}\cdot\vec{w} = \vec{0}$$

They simplify to

$$\left.\begin{aligned} a\,\|\vec{u}\|^2 &= 0 \\ b\,\|\vec{v}\|^2 &= 0 \\ c\,\|\vec{w}\|^2 &= 0 \end{aligned}\right\}$$

Since $\vec{u}, \vec{v}, \vec{w}$ are all non-zero vectors, $\|\vec{u}\|^2 \neq 0$, $\|\vec{v}\|^2 \neq 0$ & $\|\vec{w}\|^2 \neq 0$, then we imply that $a = b = c = 0$ & hence $\{\vec{u}, \vec{v}, \vec{w}\}$ is L.I.