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UNIVERSITY OF WATERLOO

MATH 137

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MIDTERM EXAM-AID

REVIEW PACKAGE

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Functions and Absolute Value

Review of Functions

A function *f*, assigns exactly one value to every element *x*. We can think of *x* as the function's input and *y* as its output. For our purposes, we can use *y* and *f(x)* interchangeably. In Calculus 1, we deal with functions taking elements of the real numbers as inputs and outputting real numbers. We are primarily concerned with using graphs as visual representations of functions.

Here are some definitions that are useful to keep in mind:

Domain: The set of elements *x* that can be inputs for a function *f*

Range: The set of elements *y* that are outputs of a function *f*

Increasing Function: A function is increasing over an interval A if for all $x \in A$, the property $x1 < x2 \rightarrow f(x1) < f(x2)$ holds.

Decreasing Function: A function is decreasing over an interval A if for all $x \in A$, the property $x1 < x2 \rightarrow f(x1) > f(x2)$ holds.

Even Function: A function with the property that for all values of *x*: $f(-x) = f(x)$

Odd Function: A function with the property that for all values of *x*: $f(-x) = -f(x)$

Note that even functions are symmetric about the y-axis and that odd functions are symmetric about the origin (0, 0).

Absolute Value

Definition:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Properties and Rules:

- $|x| = a \quad \Rightarrow \quad x = \pm\, a$
- $|x| < a \quad \Rightarrow \quad x = -a < x < a$

- $|x| > a \;\Rightarrow\; x > a$ or $x < -a$

$$|xy| = |x||y|$$

Triangle Inequality:

$$|x+y| \le |x| + |y|$$

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

Example. Given that $|x| < 2$, show that $\left|\frac{x-1}{2x^2+5}\right| \le \frac{3}{5}$

First, we split the fraction and apply the triangle inequality to obtain:

$$\left|\frac{x\ -\ 1}{2x^2+\ 5}\right| \le \left|\frac{x}{2x^2+\ 5}\right| + \left|\frac{-\ 1}{2x^2+\ 5}\right|$$

Note that $2x^2 +\ 5 \ge 5$ for any value of *x*. Therefore, if we replace the denominator with 5, we are shrinking it and thereby making the entire rational expression larger. Hence:

$$\left|\frac{x}{2x^2+\ 5}\right| + \left|\frac{-\ 1}{2x^2+\ 5}\right| \le \left|\frac{x}{5}\right| + \left|\frac{-1}{5}\right|$$

After applying properties of absolute value, we can obtain the following expression:

$$\frac{|x| +\ |-1|}{|5|}$$

By applying the definition of absolute value, we get that |-1| = 1 and |5| = 5. Since |x| < 2, as provided in the question, we can safely substitute 2 for x in the expression. Thus, we obtain:

$$\frac{|x| +\ |-1|}{|5|} = \frac{|x| +\ 1}{5} \le \frac{3}{5}$$

Sketching – The Use of Cases

Sometimes you may be asked to sketch a function that involves piecewise definitions. Generally, we start by looking for the key x-values where the function changes value. Then, we use these x-values to create different “cases”, where we analyze the function over a particular interval. We can apply similar methods to sketch implicitly-defined inequalities.

Example. Sketch the inequality $|x + y| \leq 1$.

We have two cases to consider. Our goal is to get inequalities where we isolate y.

- Case 1: $x + y \geq 0$, which implies that $\boldsymbol{y \geq -x}$
  - We have $x + y \leq 1$, or $\boldsymbol{y \leq 1 - x}$
- Case 2: $x + y < 0$, which implies that $\boldsymbol{y} < -x$
  - We have $-(x + y) \leq 1$, or $\boldsymbol{y \geq -x - 1}$

To finish, sketch each of the lines defining the y-regions and shade in their intersection.

One-to-One Functions and Inverses

One-to-One Functions

A function is **one-to-one** if it never takes the same y-value twice, that is, it satisfies:

$$f(x_1) = f(x_2) \rightarrow x_1 = x_2$$

Alternatively, we can see that a function is one-to-one if any horizontal line touches the function at most once. This is known as the **horizontal line test**. If a function is increasing and decreasing on different intervals, it cannot be one-to-one unless it is discontinuous.

Example. Identify which of the following functions are one-to-one.

- $y = \ln(x)$ [YES]
- $y = \cos(x)$ [NO]
- $y = x^n$, where n is an even integer [NO]
- $y = x^n$, where n is an odd integer [YES]

Example. Prove that $f(x) = \sqrt{x^2 + 5}$ is one-to-one on [0, ∞).

To prove this, we need to use the defintion of a one-to-one function. Set $f(x_1) = f(x_2)$.

Thus,

$$\sqrt{(x1)^2 + 5} = \sqrt{(x2)^2 + 5}$$

By squaring both sides and subtracting off the constant term, we end up with:

$$(x_1)^2 = (x_2)^2$$

Normally, by taking the square root of both sides, we would have:

$$\pm x_1 = \pm x_2$$

However, we are only considering the interval [0, ∞), therefore, *x1* and *x2* are both positive. Therefore,

$$x_1 = x_2$$

We have proved $f(x1) = f(x2) \rightarrow x1 = x2$. Hence, $f(x)$ is one-to-one on [0, ∞).

Inverse Functions

A function $f(x)$ that is one-to-one with domain A and range B has an **inverse function** $f^{-1}(x)$ with domain B and range A.

- $f^{-1}(x)$ reverses the operations of $f(x)$ in the opposite direction
- $f^{-1}(x)$ is a reflection of $f(x)$ about the line *y = x*
- $f^{-1}(a) = b \rightarrow f(b) = a$

Cancellation Identity: Let $f(x)$ and $g(x)$ be functions that are inverses of each other. Then:

$$f\big(g(x)\big) = x \qquad and \qquad g(f(x) = x$$

The cancellation identity can be applied only if x is in the domain of the inside function.

To find the equation of $f^{-1}(x)$, we first switch x and y within the equation of $f(x)$. Then, if possible, we isolate for y, or leave the function implicitly defined.

Example. Find the inverse of $f(x) = \frac{1}{1+e^x}$.

Switching x and y, we have: $x = \frac{1}{1+e^y}$.

Next, we attempt to isolate for the term containing y. Through basic algebra, we end up with:

$$e^y = \frac{1-x}{x}$$

Now, we want to get y by itself, so we take the natural logarithm of both sides. By log laws:

$$y\, ln(e) = \ln(\frac{1-x}{x})$$

Since $ln(e) = 1$, we end up with:

$$y = f^{-1}(x) = \ln(\frac{1-x}{x})$$

INVERSE TRIGONOMETRIC FUNCTIONS

Given that trigonometric functions are periodic, we know that they are not one-to-one, as they would all fail the horizontal line test. Therefore, in order to define an inverse trigonometric function, we must restrict the domain of the corresponding trigonometric function to make it one-to-one.

Definitions (by convention):

Trigonometric Function	*Restriction on the Domain*	*Inverse Trigonometric Function*	*Domain and Range of Inverse*
$y = \sin(x)$	$\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$	$y = \arcsin(x)$	Domain $-1 \leq x \leq 1$ Range $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \cos(x)$	$0 \leq x \leq \pi$	$y = \arccos(x)$	Domain $-1 \leq x \leq 1$ Range $0 \leq y \leq \pi$
$y = \tan(x)$	$\frac{-\pi}{2} < x < \frac{\pi}{2}$	$y = \arctan(x)$	Domain $x \in \mathbb{R}$ Range $\frac{-\pi}{2} < y < \frac{\pi}{2}$

Example. Solve $\cos(\arccos(x)) = \frac{1}{2}$

We know that x is an element of *arccos(x)* if $-1 \leq x \leq 1$. Therefore, we can apply the cancellation identity, and get $x = \frac{1}{2}$

Example. Evaluate $\arcsin(\sin\frac{5\pi}{4})$.

Note that $\frac{5\pi}{4}$ is not in the domain of the domain-restricted sin (x). Therefore, we cannot apply the cancellation identity.

Start with $\sin\frac{5\pi}{4}$. Since $\frac{5\pi}{4}$ falls in the third quadrant, by CAST rule, we have that $\sin\frac{5\pi}{4}$ is negative. Taking $\frac{\pi}{4}$ (the related acute angle), we can use a special triangle to give us the result $\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$.

Now, we look at $\arcsin(-\frac{1}{\sqrt{2}})$. Considering it as a function y, we have its range as $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$

Thus, we know our solution will fall in this range. We need to find a y such that $\sin(y) = -\frac{1}{\sqrt{2}}$

Within our range, we have quadrants 1 and 4. As sin is negative in quadrant 4, we can further restrict the range to $\frac{-\pi}{2} \leq y \leq 0$

The only angle that works is $\frac{-\pi}{4}$

LIMITS

EVALUATING LIMITS

Recall the basic limit laws from high school. Given that the limits exist, we have that:

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x) \quad (where\ c\ is\ a\ constant)$$

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}\quad (where\ \lim_{x\to a}g(x)\neq 0)$$

Also recall that if the left and right hand limits of a function are different, the limit does not exist.

There are two more advanced limit laws that can come in handy when evaluating limits. Assuming *n* is a positive integer, we have that:

$$\lim_{z\to a}[f(x)]^n=\left[\lim_{x\to a}f(x)\right]^n$$

$$\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a}f(x)}$$

More often, however, you will end up with limits that can't be evaluated directly by applying the laws and substituting your *a*-value. Instead, these limits will take on an indeterminate form. You must algebraically work with the function in order to get it into a form that can be easily determined. For example, factoring, rationalizing, and expanding are just some of the algebraic skills from high school you will need to recall.

Example. Evaluate $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$

By substitution, this limit has the indeterminate form $\frac{0}{0}$. Therefore, we cannot evaluate it directly.

We can rationalize the numerator by multiplying it with its conjugate. Remember to multiply the denominator as well:

$$\frac{\sqrt{1+x}\ -1}{x}\cdot\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}=\frac{x}{x(\sqrt{1+x}+1)}$$

Evaluating the limit directly by substitution, we now get that the limit is 0.

Example. Evaluate $\lim_{x \to -2} \frac{x+2}{x^3+8}$

By substitution, this limit has the indeterminate form $\frac{0}{0}$. Therefore, we cannot evaluate it directly.

Notice that the denominator factors using the sum of cubes method:

$$x^3 + 8 = (x+2)(x^2 - 2x + 4)$$

By dividing out common factors, we have:

$$\frac{x+2}{x^3+8} = \frac{x+2}{(x+2)(x^2-2x+4)} = \frac{1}{x^2-2x+4}$$

We can now evaluate our limit directly. By substitution:

$$\lim_{x \to -2} \frac{x+2}{x^3+8} = \lim_{x \to -2} \frac{1}{x^2-2x+4} = \frac{1}{12}$$

Example. Evaluate $\lim_{x \to \infty} \frac{x^3+5x}{2x^3-x^2+4}$

Obviously, we cannot evaluate this limit directly. A handy trick to work with infinity limits of rational functions is to divide each term by the highest power of *x* in the denominator (in this case, x^3):

$$\frac{x^3+5x}{2x^3-x^2+4} = \frac{\frac{x^3}{x^3} + \frac{5x}{x^3}}{\frac{2x^3}{x^3} - \frac{x^2}{x^3} + \frac{4}{x^3}} = \frac{1 + \frac{5}{x^2}}{2 - \frac{1}{x} + \frac{4}{x^3}}$$

As $x \to \infty$, $\frac{5}{x^2}$, $\frac{1}{x}$, $\frac{4}{x^3}$ all tend to 0. Therefore:

$$\lim_{x \to \infty} \frac{x^3+5x}{2x^3-x^2+4} = \frac{1}{2}$$

THE FORMAL DEFINITION OF A LIMIT

$$\lim_{x \to a} f(x) = L$$

if given any $\varepsilon > 0$, we can find a $\delta > 0$ such that:

$$0 < |x - a| < \delta \to |f(x) - L| < \varepsilon$$

A simpler way to explain the last line is: "if the distance between *x* and *a* is small, then the distance between *f(x)* and L is also small". If that statement holds true for **any** small distance, then we have our limit as L. This ties into our intuitive understanding of a limit – by making x close to a, we make *f(x)* close to L.

In order to formally prove a limit, we need to use this epsilon-delta definition. Unlike traditional proofs, we must start with the converse of the statement Q => P

Example. Prove that $\lim_{x \to 2} 5x - 6 = 4$

Set $\varepsilon > 0$

By the definition of a limit, we set up our inequality involving the difference between our function and the limit:

$$|(5x - 6) - 4| < \varepsilon$$

Next, through basic algebra, we try to arrive at the inequality involving *x* and *a*:

$$|5x - 10| < \varepsilon$$
$$|5||x - 2| < \varepsilon$$
$$|x - 2| < \frac{\varepsilon}{5}$$

Select $\delta = \frac{\varepsilon}{5}$

Now, by substituting our value for delta, we can directly prove the limit by starting with $0 < |x - a| < \delta$ and arriving at $|f(x) - L| < \varepsilon$

■

SQUEEZE THEOREM

$$If\ g(x) \leq f(x) \leq h(x)\ and\ \lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L\,, then\ \lim_{x \to a} f(x) = L$$

Example. Evaluate $\lim_{x \to 0} \sqrt[2]{x^3} \cdot \cos(\frac{1}{x^2})$

We know, by the definition of the cosine function:

$$-1 \leq \cos\left(\frac{1}{x^2}\right) \leq 1$$

Multiplying across by $\sqrt[2]{x^3}$:

$$-\sqrt[2]{x^3} \leq \sqrt[2]{x^3} \cdot \cos\left(\frac{1}{x^2}\right) \leq \sqrt[2]{x^3}$$

Taking the limits:

$$\lim_{x \to 0} -\sqrt[2]{x^3} \leq \lim_{x \to 0} \sqrt[2]{x^3} \cdot \cos\left(\frac{1}{x^2}\right) \leq \lim_{x \to 0} \sqrt[2]{x^3}$$

$$0 \leq \lim_{x \to 0} \sqrt[2]{x^3} \cdot \cos\left(\frac{1}{x^2}\right) \leq 0$$

Thus, by squeeze theorem, our limit is 0.
A handy result of the squeeze theorem is the Fundamental Trigonometric Limit:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

which can be used to evaluate complex limits involving trigonometric functions.

Example. Evaluate $\lim_{x \to 0} \frac{\tan(x)}{x}$

Recall a trigonometric identity:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Applying this to our function and dividing, we have:

$$\frac{\tan(x)}{x} = \frac{\sin(x)}{x \cdot \cos(x)} = \frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)}$$

Substituting directly, we can evaluate the limit as:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)} = 1 \cdot 1 = 1$$

CONTINUITY

THEOREMS OF CONTINUITY

A function $f(x)$ is continuous at a point $x = a$ if $\lim_{x \to a} f(x) = f(a)$.
A function is continuous over an interval A if it is continuous on every x in A.

Example. Find the values for *b* and *c* such that the function is continuous at x = 2.

$$f(x) = \begin{cases} x^2 + 1 & x > 2 \\ c & x = 2 \\ x + b & x > 2 \end{cases}$$

Observe the one-sided limits around *x* = 2. We need $\lim_{x \to 2+} f(x) = \lim_{x \to 2-} f(x)$.
Setting them to equal each other, we have:

$$\lim_{x \to 2+} x + b = \lim_{x \to 2-} x^2 + 1$$
$$2 + b = 5$$
$$b = 3$$

With this value of b, we have $\lim_{x \to 2+} f(x) = \lim_{x \to 2-} f(x) = \lim_{x \to 2} f(x) = 5$

Now all we need is for the limit to equal the function at *x* = 2. Thus, we set *c* = 5.
If $f(x)\ and\ g(x)$ are continuous functions at $x = a$, then:

- $f(x) + g(x)$ is continuous at $x = a$
- $f(x)g(x)$ is continuous at $x = a$
- $\frac{f(x)}{g(x)}$ is continuous at $x = a$ (given that $g(a) \neq 0$)
- If $g(x)$ is continuous at $x = a$ and $f(x)$ is continuous at $g(a)$, then $f(g(x))$ is continuous at $x = a$

Note that all polynomial, exponential, logarithmic, trigonometric, and reciprocal functions are continuous **on their domains**.

Example. Identify where $f(x) = e^{x^2 + \sin(x)}$ is continuous.

$x^2 \text{and } \sin(x)$ are both continuous on all real numbers *x*. Thus, by continuity theorems, their sum is also continous. e^x is continuous on all real numbers *x*. Therefore, by continuity theorems, its composition with $x^2 + \sin(x)$ is also continuous on all real numbers.

Types of Discontinuities

- Infinite
    - When a function has a vertical asymptote
- Jump
    - When the one-sided limits do not equal one another
- Removable
    - When the limit does not equal the function value at a point

- Infinite Oscillations
    - When there are an infinite number of oscillations in a neighbourhood of a point
    - EX: $f(x) = \sin\left(\frac{1}{x}\right)$

INTERMEDIATE VALUE THEOREM

If a function $f(x)$ is continuous for all x in an interval $[a, b]$ and $f(a) > 0$ and $f(b) < 0$ (or vice versa), then there exists a point $c \in (a, b)$ such that $f(c) = 0$.

Example. Show that $f(x) = x^4 - 4x^3 + 1$ has a root between 0 and 1.

$f(x)$ is a polynomial function, so it is continuous on all $x \in R$
$f(0) = 1$
$f(1) = -2$

Thus, by the IVT, the function crosses the x-axis between 0 and 1.

The IVT is an existence theorem, in that it allows us to prove the existence of a root. The **Method of Bisection** is an algorithm that allows us to approximate the location of the root.

Example. Approximate the root of $f(x) = x^4 - 4x^3 + 1$ between 0 and 1 to $\frac{1}{8}$ accuracy.

We start by taking the midpoint of the interval containing the root, in this case $x = 0.5$. Now, evaluating the function at this point, we get $f(0.5) = 0.5625$.

Since the function is decreasing on the interval, that means at $x = 0.5$, it has not reached its root yet. Thus, we can narrow our interval to $(0.5, 1)$.

Now we repeat our algorithm by taking the midpoint of $(0.5, 1)$, which is 0.75. Note that this point approximates the root with an accuracy of $\frac{1}{4}$ or 0.25.

$f(0.75) = -0.37$, meaning we have gone past the root. Thus, we can narrow our interval again to $(0.5, 0.75)$.

The midpoint of the interval is 0.625, which gives us an accuracy of $\frac{1}{8}$, as required.

DIFFERENTIAL CALCULUS

DEFINITION OF THE DERIVATIVE

There are two limit-based derivative definitions we can use, sometimes referred two as **difference quotients** or **first principles**:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

The second definition is also known as **Newton's quotient**. Both give the derivative of a function at a point, which geometrically refers to the slope of the tangent line to the function at that point.

In order to evaluate the derivative at a point using first principles, some skill with limits will be needed.

DIFFERENTIABILITY

In single-variable calculus, the **differentiability** of a function at a point refers to the existence of the derivative at that point.

Example. Determine if $f(x) = \begin{cases} x^5 \sin\left(\frac{3}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at x = 0.

Because this function is defined piecewise, we cannot use our derivative rules. We must use the derivative definition. Whether the function is differentiable or not depends on the existence of the limit.

Applying Newton's quotient:

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$$
$$f'(0) = \lim_{x\to 0}\frac{f(x)}{x}$$

Note that we are interested in the limit as x **approaches** 0 – that is to say, it does not **equal** 0. Therefore:

$$f'(0) = \lim_{x\to 0} \frac{x^5 \sin(\frac{3}{x})}{x} = \lim_{x\to 0} x^4 \sin(\frac{3}{x})$$

Now, by applying squeeze theorem, we can easily see that the limit is 0. Therefore, the function is differentiable at x = 0.

Theorem. If a function is differentiable at a point, it is also continuous at that point.

This is a very important theorem. Sometimes it is easier to show continuity at a point by proving differentiability at that point. Note that by the Contrapositive Law from MATH 135, we also have the statement: **"If a function is NOT continuous at a point, then it is NOT differentiable at the point**."

Also it is important to note that this is not an "if and only if" theorem. Thus, the converse of the theorem, "If a function is continuous at a point, it is also differentiable at that point" is **FALSE!**

Derivative Rules

Power Rule.

$$If\ f(x) = ax^n, then\ f'(x) = nax^{n-1}$$

Product Rule.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

Quotient Rule.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Chain Rule.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example. Find the derivative of $f(x) = (2x+3)^5$

Note that this function can be rewritten as a composition $f(x) = u^5$, where $u = 2x + 3$.

Therefore, by Chain Rule, we have:

$$f'(x) = 5u^4 \cdot 2 = 10(2x+3)^4$$

The chain rule in this case is referred to as the **power of a function** rule. It is used when the exponent applies not just to a single *x*, but to an entire function.

Exponential Derivatives.

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = a^x \cdot \ln(a)$$

Note: If *e* or *a* are raised to some function (ie. not just *x* on its own), then we must apply Chain Rule

Example. Find the equation of the tangent to $f(x) = e^x x^3$ at $x = 1$

Using Product Rule:

$$f'(x) = e^x x^3 + e^x(3x^2)$$

At x = 1:

$$f'(1) = e(1) + e(3) = 4e$$

Thus, the slope of the tangent at x = 1 is 4*e*. At x = 1, we have $f(1) = e$. Therefore, in slope-point form, the equation of the tangent is:

$$y - e = 4e(x-1)$$

Trigonometric Derivatives.

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\sec(x) = \tan(x)\sec(x)$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

Note: If a trig function is applied to some function (ie. not just *x* on its own), then we must apply Chain Rule

Example. Find the points where the tangent to $f(x) = \frac{\sin(x)}{2+\cos(x)}$ is horizontal.

A horizontal tangent has a slope of 0.
Using quotient rule:

$$f'(x) = \frac{\cos(x)\,(2+\cos(x)) - \sin(x)(-\sin(x))}{(2+\cos(x))^2}$$

$$f'(x) = \frac{2\cos(x) + \cos^2(x) + \sin^2(x)}{(2+\cos(x))^2}$$

By the Pythagorean trig identity:

$$f'(x) = \frac{2\cos(x) + 1}{(2+\cos(x))^2}$$

Setting $f'(x) = 0$, we have:

$$2\cos(x) + 1 = 0$$

$$\cos(x) = \frac{-1}{2}$$

$$x = \frac{2\pi}{3} + 2\pi n \quad (n \in Z)$$

Hyperbolic Derivative

We know by definition:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

As you may expect, the derivatives of the hyperbolic functions are very similar to the trigonometric functions:

$$\frac{d}{dx}\sinh(x) = \cosh(x)$$

$$\frac{d}{dx}\cosh(x) = -\sinh(x)$$

$$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$$

PROOFS

LIMIT SUM LAW

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a}[f(x) + g(x)] = L + M$

Proof

Let $\varepsilon > 0$ be given.
If $0 < |x - a| < \delta$, then $|f(x) + g(x) - (L + M)| < \varepsilon$

By Triangle Inequality:

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M|$$

$|f(x) + g(x) - (L + M)| < \varepsilon$ if $|f(x) - L| < \frac{\varepsilon}{2}$ and $|g(x) - M| < \frac{\varepsilon}{2}$

Then, there exist $\delta 1\ and\ \delta 2$ such that:

If $0 < |x - a| < \delta 1$, then $|f(x) - L| < \frac{\varepsilon}{2}$

If $0 < |x - a| < \delta 2$, then $|g(x) - M| < \frac{\varepsilon}{2}$

Let $\delta = \min(\delta_1, \delta_2)$
Thus, if $0 < |x - a| < \delta$, then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$

Therefore, $|f(x) + g(x) - (L + M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
Hence, $\lim_{x\to a}[f(x) + g(x)] = L + M$

■

DIFFERENTIABILITY IMPLIES CONTINUITY

If a function is differentiable at a point, it is also continuous at that point.

Proof
For *x* close to a point *a*, we have:

$$f(x) = \frac{f(x) - f(a)}{x - a}(x - a) + f(a)$$

Taking limits, we have:

$$\lim_{z\to a} f(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x\to a}(x - a) + \lim_{x\to a} f(a)$$

$$\lim_{z\to a} f(x) = f'(a) \cdot (0) + f(a)$$

$$\lim_{z\to a} f(x) = f(a)$$

Therefore, $f(x)$ is continuous at $x = a$

■

PRODUCT RULE

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

Proof
Using first principles:

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Adding and subtracting $f(x+h)g(x)$ in the numerator:

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) - f(x)g(x) + f(x+h)g(x)}{h}$$

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0} f(x+h) \cdot \frac{[g(x+h) - g(x)]}{h} + g(x)\frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x) \quad ■$$