1A.

	B	$\bar{B}$	
A	$3/12$	$4/12$	$7/12$
$\bar{A}$	$2/12$	$3/12$	$5/12$
	$5/12$	$7/12$	

MARGINAL : $P(A) = P(A \cap B) + P(A \cap \bar{B}) = \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$

1B. $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{3/12}{5/12} = \dfrac{3}{5}$

1C. $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{3/12}{7/12} = \dfrac{3}{7}$

1D. $P(A \cap B) = \frac{3}{12} \neq \underbrace{\frac{35}{144}}_{P(A) \cdot P(B)}$ THEREFORE, A AND B ARE NOT INDEPENDENT.

2A. ~~[illegible]~~ ~~[illegible]~~ ~~$S = \{0, k, 10\}$~~.

NOTE THAT 10 IS MAXIMUM VALUE X CAN TAKE.

~~2B.~~ THEREFORE, $F(10) = 1$ FOR $P(\cdot)$ TO BE A P.D.F.

2B. $\sum_{x \in S} P(x) = 1 \implies P(0) + P(k) + P(10) = 1$

$$P(k) = 1 - P(0) - P(10)$$

$$= 1 - 1/10 - 2/10 = 7/10.$$

$$\implies c = P(k) = 7/10.$$

2C. ~~[illegible]~~

$$\mu_X = E[X] = \sum_{x \in S} x P(x) = 0 \times \frac{1}{10} + k \times \frac{7}{10} + 10 \times \frac{1}{5} = 4.$$

$$\implies \frac{k \cdot 7}{10} + 2 = 4$$

$$\implies \frac{7k}{10} = 2 \implies 7k = 20$$

$$\implies k = \frac{20}{7}.$$

2D. $\sigma_X^2 = E[X^2] - \mu_X^2$

$$E[X^2] = \sum_{x \in S} x^2 P(x) = 0^2 \times \frac{1}{10} + \frac{20^2}{7^2} \cdot \frac{7}{10} + 10^2 \cdot \frac{1}{5}$$

$$= \frac{20^2}{70} + 20 = \frac{400}{70} + 20.$$

$$\sigma_X^2 = \frac{40}{7} + 20 - \underbrace{16}_{4^2} = \frac{40}{7} + 4 = \frac{40 + 28}{7} = \frac{68}{7}. = 9.714.$$

3A. $\mu_Y = E[Y] = E\left[\frac{X+3}{5}\right]$.

WE KNOW $E[g(x)] = \sum_x g(x)P(x)$.

SO, IN THIS CASE, $g(X) = \frac{X+3}{5}$.

$$E\left[\frac{X+3}{5}\right] = \sum_x \left(\frac{X+3}{5}\right)P(x) = \frac{1}{5}\sum(x+3)P(x)$$

$$= \frac{1}{5}\left[\underbrace{\sum x P(x)}_{\mu_X} + 3\underbrace{\sum P(x)}_{1}\right]$$

$$= \frac{\mu_X}{5} + \frac{3}{5}.$$

3B. $\sigma_Y^2 = E\left[(Y-\mu_Y)^2\right]$

$$= E\left[\left(\frac{X+3}{5} - \left(\frac{\mu_X}{5} + \frac{3}{5}\right)\right)^2\right]$$

$$= E\left[\left(\frac{X}{5} - \frac{\mu_X}{5}\right)^2\right] = E\left[\frac{1}{25}(x-\mu_x)^2\right]$$

$$= \frac{1}{25}E\left[(X-\mu_X)^2\right]$$ B/C OF LINEARITY OF $E[\cdot]$, SHOWN ABOVE.

$$= \frac{\sigma_X^2}{25}$$

~~3C. E[2X + 50Y] = E[2X] + E[50Y] = 2μx + 50 E[Y]~~

~~= 2μx + 50 E[(μx+3)/5]~~

~~= 2μx + 10μx + 30~~

~~= 12μx + 30.~~

~~3D. E[2X + σ²x] = E[2X] + E[σ²x] = E[2X] BECAUSE σ²x IS A CONSTANT~~

~~= 2μx.~~

3C. $\mathrm{Cov}(X,Y) = E\left[(X-\mu_X)(Y-\mu_Y)\right]$

$$= E\left[(X-\mu_X)\left(\underbrace{\frac{X+3}{5}}_{Y} - \underbrace{\frac{\mu_X+3}{5}}_{\mu_Y}\right)\right]$$

$$= E\left[(X-\mu_X)\left(\frac{X-\mu_X}{5}\right)\right]$$

$$= E\left[\frac{1}{5}(X-\mu_X)^2\right] = \frac{1}{5}\sigma_X^2.$$

3D. $$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{\frac{1}{5}\sigma_X^2}{\sigma_X \cdot \frac{1}{5}\sigma_X} = 1.$$

3E. $$E\left[2X + 3\sigma_X^2\right] = E\left[2X\right] + E\left[3\sigma_X^2\right]$$

$$= 2\mu_X + 3\sigma_X^2.$$

3F. $$\mathrm{VAR}(2X + 3\sigma_X^2) = E\left[\left(2X + 3\sigma_X^2 - E[2X+3\sigma_X^2]\right)^2\right]$$

$$= E\left[\left(2(X-\mu_X)^{*} + (3\sigma_X^2 - E[3\sigma_X^2])\right)^2\right].$$

* NOTE THAT σ_X^2 IS A CONSTANT, SO $E[\sigma_X^2] = \sigma_X^2$.

$$\Rightarrow E\left[\left[2(X-\mu_X)\right]^2\right] = 4\sigma_X^2.$$

4. $P(Y=\cancel{10}\ 0) = P(X=0 \cup X=1)$ $\quad \lambda = 4/6 = 2/3.$

$P(X=0) =$

$$\frac{(2/3)^0 e^{-2/3}}{0!} = e^{-2/3} =$$

$$P(X=1) = \frac{(2/3)^1 e^{-2/3}}{1!} = \frac{2}{3} e^{-2/3}$$

$$P(X \le 1) = \left(1+\frac{2}{3}\right) e^{-2/3} = 0.8557.$$

So $P(Y=\cancel{2}0) = 0.8557.$

4A. $P(Y=40) = 1 - P(Y=\cancel{1}0) = 0.14430$

$$E[Y] = \sum_Y y P(y) = \cancel{3}0 \cdot P(\cancel{2}0) + 40 \cdot P(40)$$

$$= \cancel{22.8861} \quad 5.7722.$$

4B. $Var(Y)$ ~~m~~ $= ?$ NOTE THAT Y IS LIKE BERNOULLI w/ $P(1) = 0.14430.$

LET $Y = 40 \cdot Z$ WHERE Z IS BERNOULLI w/ $P(1) = 0.14430$

$$\sigma_Y^2 = 40^2 \sigma_Z^2 = 40^2 \cdot \pi \cdot (1-\pi)$$

$$= 40^2 \cdot (0.14430)(0.8557) = 197.5695$$

5. PRIOR BELIEF: $P(W) = 1/2$. SO $P(\bar{W}) = 1/2$ IS PROB. OF THE COIN BEING FAIR.

DENOTE M THE AMOUNT I REQUIRE TO BREAK EVEN.

THE PROBABILITY OF $\underline{1}$ (A QUEENIE) IS:
~~MY EXPECTED RETURN IS~~

5A.

$$P(1) = \cancel{P(1)}\ P(1 \mid W) \cdot P(W) + P(1 \mid \bar{W}) \cdot P(\bar{W})$$

SO MY EXPECTED RETURN GIVEN \$M PER QUEENIE IS

$E[MX]$ WHERE $X = 1$ WITH THE ABOVE PROBABILITY.

$$= M\left[P(1 \mid W)P(W) + P(1 \mid \bar{W})P(\bar{W})\right]$$

$$= M\left[\frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}\right]$$

$$= M\left[\frac{1}{8} + \frac{1}{4}\right] = \frac{M \cdot 3}{8}.$$

I BREAK EVEN IN EXPECTATION IFF $E[\$] \geq 2$, OR

$$\frac{3M}{8} \geq 2 \implies M \geq \frac{16}{3}.$$

SO $\$\frac{16}{3}$ IS MINIMUM AMOUNT REQUIRED GIVEN PRIOR PROBABILITY.

5B. $$P(W \mid \overset{\text{LOONIE}}{0}) = \frac{P(0 \mid W)P(W)}{P(0 \mid W)P(W) + P(0 \mid \bar{W})P(\bar{W})} = \frac{3/4 \cdot 1/2}{3/4 \cdot 1/2 + 1/2 \cdot 1/2} = \frac{3/8}{3/8 + 1/4} = \frac{3/8}{5/8} = 3/5.$$

WE THINK IT'S MORE LIKELY NOW THAT THE COIN IS WEIGHTED.

1ST FLIP LOONIE.

5C. $$P(1 \mid 0) = P(1 \mid W) \cdot P(W \mid 0) + P(1 \mid \bar{W})P(\bar{W} \mid 0)$$

$$= 1/4 \cdot 3/5 + 1/2 \cdot 2/5 = \frac{3}{20} + \frac{4}{20} = \frac{7}{20}.$$

$P(\text{QUEENIE}) \implies \boxed{\frac{13}{20} \quad P(\text{LOONIE} \mid \text{DATA})}$

5D. $$\underbrace{\frac{M \cdot 7}{20} \geq 2}_{E[\$] \geq 2.} \qquad M \geq \frac{40}{7}.$$

SO $\$\frac{40}{7}$ IS MINIMUM AMOUNT REQUIRED TO PLAY ONE MORE FLIP.

5E. $P(W \mid X=4) = \dfrac{P(X=4 \mid W)\,P(W)}{P(X=4 \mid W)\,P(W) + P(X=4 \mid \bar{W})\,P(\bar{W})}$

WHERE X CAN HAVE "SUCCESS" PROB OF $3/4$ IF WEIGHTED (W) OR $1/2$ IF FAIR $(\bar{W})$

$P(X=4 \mid W) = C^5_4 \left(\frac{3}{4}\right)^4 \left(\frac{1}{4}\right)^1 = 0.3955$

$P(X=4 \mid \bar{W}) = C^5_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1 = 0.15625$

GIVING US $P(W \mid X=4) = \dfrac{0.3955 \times \overset{\text{PRIOR } P(W)}{1/2}}{0.3955 \times 1/2 + 0.15625 \times \underset{\text{PRIOR } P(W)}{1/2}} = 0.7168$

COMBINE THIS TO GIVE $P(\text{LOONIE} \mid X=4) = P(\text{LOONIE} \mid W)\,P(W \mid X=4) + P(\text{LOONIE} \mid \bar{W})\,P(\bar{W} \mid X=4)$

$= \frac{3}{4} \cdot 0.7168 + \frac{1}{2} \cdot (1 - 0.7168) = 0.6792$

Midterm 1

Econ 2222A

October 16, 2013, 7-9pm

Instructions:

- Show your work
- Express numeric answers in either decimal form or simplified (that is, "reduced") fraction form. Non-reduced fractions may be marked as incorrect.
- Use of only non-graphic (i.e. non-programmable) calculators is allowed.
- All exam materials, INCLUDING THIS SHEET OF PAPER, must be turned in with the exam.

Points breakdown:

Question	1	2	3	4	5	Total
Points	12	18	26	11	33	100

Useful formulae:

Shortcut formula for variance of RV X: $\sigma_X^2 = E[X^2] - (E[X])^2$

If X is Bernoulli RV with prob. of success π, then $P(X = 1) = \pi$ and $P(X = 0) = 1 - \pi$.

If X is Binomial with prob. of success π for n independent Bernoulli trials, then $f(x) = C_x^n \pi^x (1 - \pi)^{n-x}$

If X is Poisson with expected successes per interval λ then $f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$, where $e = 2.718...$

1. (12 points) Event A occurs if and only if I rode my bicycle to the university. Event B occurs if and only if it rained that day. The below table indicates how events A and B relate to each other. Each cell is the probability of the intersection of the events in the corresponding row and column.

	B	$\bar{B}$
A	1/4	1/3
$\bar{A}$	1/6	1/4

1a. (3 points) What is the probability that I rode my bicycle to the university?

1b. (3 points) What is the probability that I rode my bicycle to the university, given that it has rained?

1c. (3 points) What is the probability that it rained that day, given that I rode my bicycle to the university?

1d. (3 points) Are events A and B independent? Why or why not?

2. (18 points) X is a discrete random variable with probability distribution function $P(\cdot)$:

$$P(0) = 1/10$$

$$P(k) = c$$

$$P(10) = 1/5$$

Where c is a constant and $k \in (0, 10)$ is a real number (not necessarily an integer).

2a. (3 points) What is the cumulative distribution function, $F(x_0)$, evaluated at $x_0 = 10$?

2b. (5 points) What value of c makes P a valid probability distribution function?

2c. (5 points) Assume c is what you found in part (2b). What value of k makes the mean of $X = 4$?

2d. (5 points) Assume c and k are what you found in parts (2b) and (2c). What is the variance of X?

3. (26 points) X is a discrete random variable with probability distribution function $P(x)$, mean μ_X, and variance σ_X^2. Let Y be a discrete random variable defined according to $Y = \frac{X+3}{5}$.

3a. (4 points) What is μ_Y?

3b. (4 points) What is σ_Y^2?

3c. (5 points) What is $\text{Cov}(X, Y)$?

3d. (5 points) What is ρ_{XY} (the correlation between X and Y)?

3e. (4 points) What is $\text{E}\left[2X + 3\sigma_X^2\right]$?

3f. (4 points) What is $\text{Var}(2X + 3\sigma_X^2)$?

4. (11 points) Let X be a Poisson random variable denoting the expected number of buses to arrive at the bus stop. We expect 4 buses to arrive per hour. Let Y be a random variable that is 0 if at most one arrives in the next 10 minutes, and 40 if two or more buses arrive in the next 10 minutes.

4a. (5 points) What is the mean of Y?

4b. (6 points) What is the variance of Y?

5. (33 points) You're in a big city and you need some money. You come across what seems to be a fantastic opportunity – a guy in a trenchcoat wearing a hat with a peacock feather tucked into the hatband who will pay you money if a coin flip comes up Queenie. He charges you \$2 per coin flip.

Each flip can result in either a Loonie or a Queenie coming face-up. You don't know the true probability that the coin will come up Loonie, but you do know that the coin flips are independent, given the true probability of success.

You also know that the coin is either fair ($P(\text{Loonie}) = \frac{1}{2}$) or that it's weighted such that $P(\text{Loonie}) = \frac{3}{4}$. Let W denote the event that the coin is weighted. Your prior belief is that there's an even chance that the coin is fair or weighted, i.e. $P(W) = \frac{1}{2}$.

5a. (6 points) Given your prior belief about the probability the coin is fair, what is the minimum amount he would have to pay you so that in expectation you would not lose money when the next flip is realized?

5b. (6 points) The first coin toss comes up Loonie. What is your belief now that the coin is weighted? How does this relate to your prior probability of the coin being weighted, and why?

5c. (6 points) Given the result of the first coin toss, what is your belief about the probability that the next coin flip will come up Loonie? I prefer numeric answers!

5d. (5 points) Given the result of this first coin toss, what is the minimum amount he would have to pay you so that in expectation you would not lose any more money when the next flip is realized? (Treat the \$2 paid for the first flip as a sunk cost – that is, it does not enter your calculation for this next flip. This is what economists do!)

5e. (10 points) There are four more coin tosses, and it turns out that three of them have come up Loonie. Including the first toss, this means that so far there have been five coin tosses, four of which have resulted in a Loonie. What is your belief about the probability that the next flip will be Loonie? I prefer numeric answers!