

MATH 3007A - Functions of a Complex Variable

Solution Set 2

1. Determine $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$, where

(a) $\theta = 0 \Rightarrow \sin(\theta) = 0, \cos(\theta) = 1$ and $\tan(\theta) = 0$.

(b) $\theta = \frac{\pi}{6} \Rightarrow \sin(\theta) = \frac{1}{2}, \cos(\theta) = \frac{\sqrt{3}}{2}$ and $\tan(\theta) = \frac{1}{\sqrt{3}}$.

(c) $\theta = \frac{\pi}{4} \Rightarrow \sin(\theta) = \frac{1}{\sqrt{2}}, \cos(\theta) = \frac{1}{\sqrt{2}}$ and $\tan(\theta) = 1$.

(d) $\theta = \frac{\pi}{3} \Rightarrow \sin(\theta) = \frac{\sqrt{3}}{2}, \cos(\theta) = \frac{1}{2}$ and $\tan(\theta) = \sqrt{3}$.

(e) $\theta = \frac{\pi}{2} \Rightarrow \sin(\theta) = 1, \cos(\theta) = 0$ and $\tan(\theta)$ is undefined.

(f) $\theta = \frac{2\pi}{3} \Rightarrow \sin(\theta) = \frac{\sqrt{3}}{2}, \cos(\theta) = -\frac{1}{2}$ and $\tan(\theta) = -\sqrt{3}$.

(g) $\theta = \frac{3\pi}{4} \Rightarrow \sin(\theta) = \frac{1}{\sqrt{2}}, \cos(\theta) = -\frac{1}{\sqrt{2}}$ and $\tan(\theta) = -1$.

(h) $\theta = \frac{5\pi}{6} \Rightarrow \sin(\theta) = \frac{1}{2}, \cos(\theta) = -\frac{\sqrt{3}}{2}$ and $\tan(\theta) = -\frac{1}{\sqrt{3}}$.

(i) $\theta = \pi \Rightarrow \sin(\theta) = 0, \cos(\theta) = -1$ and $\tan(\theta) = 0$.

(j) $\theta = \frac{7\pi}{6} \Rightarrow \sin(\theta) = -\frac{1}{2}, \cos(\theta) = -\frac{\sqrt{3}}{2}$ and $\tan(\theta) = \frac{1}{\sqrt{3}}$.

(k) $\theta = \frac{5\pi}{4} \Rightarrow \sin(\theta) = -\frac{1}{\sqrt{2}}, \cos(\theta) = -\frac{1}{\sqrt{2}}$ and $\tan(\theta) = 1$.

(l) $\theta = \frac{4\pi}{3} \Rightarrow \sin(\theta) = -\frac{\sqrt{3}}{2}, \cos(\theta) = -\frac{1}{2}$ and $\tan(\theta) = \sqrt{3}$.

(m) $\theta = \frac{3\pi}{2} \Rightarrow \sin(\theta) = -1, \cos(\theta) = 0$ and $\tan(\theta)$ is undefined.

(n) $\theta = \frac{5\pi}{3} \Rightarrow \sin(\theta) = -\frac{\sqrt{3}}{2}, \cos(\theta) = \frac{1}{2}$ and $\tan(\theta) = -\sqrt{3}$.

(o) $\theta = \frac{7\pi}{4} \Rightarrow \sin(\theta) = -\frac{1}{\sqrt{2}}, \cos(\theta) = \frac{1}{\sqrt{2}}$ and $\tan(\theta) = -1$.

(p) $\theta = \frac{11\pi}{6} \Rightarrow \sin(\theta) = -\frac{1}{2}, \cos(\theta) = \frac{\sqrt{3}}{2}$ and $\tan(\theta) = -\frac{1}{\sqrt{3}}$.

(q) $\theta = 2\pi \Rightarrow \sin(\theta) = 0, \cos(\theta) = 1$ and $\tan(\theta) = 0$.

(r) $\theta = -\frac{\pi}{6} \Rightarrow \sin(\theta) = -\frac{1}{2}, \cos(\theta) = \frac{\sqrt{3}}{2}$ and $\tan(\theta) = -\frac{1}{\sqrt{3}}$.

(s) $\theta = -\frac{\pi}{4} \Rightarrow \sin(\theta) = -\frac{1}{\sqrt{2}}, \cos(\theta) = \frac{1}{\sqrt{2}}$ and $\tan(\theta) = -1$.

(t) $\theta = -\frac{\pi}{3} \Rightarrow \sin(\theta) = -\frac{\sqrt{3}}{2}, \cos(\theta) = \frac{1}{2}$ and $\tan(\theta) = -\sqrt{3}$.

(u) $\theta = -\frac{\pi}{2} \Rightarrow \sin(\theta) = -1, \cos(\theta) = 0$ and $\tan(\theta)$ is undefined.

(v) $\theta = -\frac{2\pi}{3} \Rightarrow \sin(\theta) = -\frac{\sqrt{3}}{2}, \cos(\theta) = -\frac{1}{2}$ and $\tan(\theta) = \sqrt{3}$.

(w) $\theta = -\frac{3\pi}{4} \Rightarrow \sin(\theta) = -\frac{1}{\sqrt{2}}, \cos(\theta) = -\frac{1}{\sqrt{2}}$ and $\tan(\theta) = 1$.

- (x) $\theta = -\frac{5\pi}{6} \Rightarrow \sin(\theta) = -\frac{1}{2}, \cos(\theta) = -\frac{\sqrt{3}}{2}$ and $\tan(\theta) = \frac{1}{\sqrt{3}}$.
 (y) $\theta = -\pi \Rightarrow \sin(\theta) = 0, \cos(\theta) = -1$ and $\tan(\theta) = 0$.
 (z) $\theta = -\frac{727\pi}{6} \Rightarrow \sin(\theta) = \frac{1}{2}, \cos(\theta) = -\frac{\sqrt{3}}{2}$ and $\tan(\theta) = -\frac{1}{\sqrt{3}}$.

2. Express z in the polar form $re^{i\theta}$, $r > 0$, $\theta \in \mathbb{R}$, where

- (a) $z = \sqrt{3} - i \Rightarrow |z| = 2 \Rightarrow z = 2(\frac{\sqrt{3}}{2} - \frac{1}{2}i) \Rightarrow z = 2e^{-\pi i/6}$.
 (b) $z = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i \Rightarrow |z| = 3 \Rightarrow z = 3(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) \Rightarrow z = 3e^{2\pi i/3}$.
 (c) $z = -3\sqrt{3} - 3i \Rightarrow |z| = 6 \Rightarrow z = 6(-\frac{\sqrt{3}}{2} - \frac{1}{2}i) \Rightarrow z = 6e^{7\pi i/6}$.
 (d) $z = -i \Rightarrow |z| = 1 \Rightarrow z = e^{-\pi i/2}$.
 (e) $z = 1 + \sqrt{3}i \Rightarrow |z| = 2 \Rightarrow z = 2(\frac{1}{2} + \frac{\sqrt{3}}{2}i) \Rightarrow z = 2e^{\pi i/3}$.
 (f) $z = i - \sqrt{3} \Rightarrow |z| = 2 \Rightarrow z = 2(-\frac{\sqrt{3}}{2} + \frac{1}{2}i) \Rightarrow z = 2e^{5\pi i/6}$.
 (g) $z = -1 - i \Rightarrow |z| = \sqrt{2} \Rightarrow z = \sqrt{2}(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i) \Rightarrow z = \sqrt{2}e^{-3\pi i/4}$.

3. Let $z = 2e^{\frac{2\pi}{3}}$ and $w = 3e^{-\frac{\pi}{6}}$.

- (a) What is the effect on z of multiplication of z by w ?

Solution:

z is rotated clockwise by $\frac{\pi}{6}$ and its magnitude is multiplied by 3.

- (b) What is the effect on w of multiplication of w by z ?

Solution:

w is rotated counterclockwise by $\frac{2\pi}{3}$ and its magnitude is multiplied by 2.

4. Express $\cos(4\theta)$ and $\sin(4\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

Solution:

By De Moivre's formula, $\cos(4\theta) + i\sin(4\theta) = e^{4i\theta} = (e^{i\theta})^4 = [\cos(\theta) + i\sin(\theta)]^4$
 $= \cos^4(\theta) + 4i\cos^3(\theta)\sin(\theta) - 6\cos^2(\theta)\sin^2(\theta) - 4i\cos(\theta)\sin^3(\theta) + \sin^4(\theta) \Rightarrow$
 $\cos(4\theta) = \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)$ and
 $\sin(4\theta) = 4\cos^3(\theta)\sin(\theta) - 4\cos(\theta)\sin^3(\theta)$.

5. Find the cube roots of 27 and express them in the form $a + bi$, $a, b \in \mathbb{R}$.

Solution:

$27 = 27e^{2\pi ik} \Rightarrow 27^{1/3} = 3e^{2\pi ik/3}$, $k = 0, 1, 2$. Hence, the cube roots of 27 are
 $3, 3e^{2\pi i/3} = 3(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)$, and $3e^{4\pi i/3} = 3(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)$.

6. Find the sixth roots of -8 and express them in the form $a + bi$, $a, b \in \mathbb{R}$.

Solution:

$-8 = 8e^{\pi i + 2\pi ik} \Rightarrow (-8)^{1/6} = 8^{1/6}e^{\pi i(1+2k)/6}$, $k = 0, 1, 2, 3, 4, 5$. Hence, the sixth roots of -8 are $8^{1/6}e^{\pi i/6} = \sqrt{2}(\frac{\sqrt{3}}{2} + \frac{1}{2}i)$, $8^{1/6}e^{3\pi i/6} = \sqrt{2}i$, $8^{1/6}e^{5\pi i/6} = \sqrt{2}(-\frac{\sqrt{3}}{2} + \frac{1}{2}i)$,
 $8^{1/6}e^{7\pi i/6} = \sqrt{2}(-\frac{\sqrt{3}}{2} - \frac{1}{2}i)$, $8^{1/6}e^{9\pi i/6} = -\sqrt{2}i$, and $8^{1/6}e^{11\pi i/6} = \sqrt{2}(\frac{\sqrt{3}}{2} - \frac{1}{2}i)$.

7. Find the fourth roots of $8 + 8\sqrt{3}i$. You may leave them in polar form.

Solution:

$8 + 8\sqrt{3}i = 16(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 16e^{\pi i/3 + 2\pi ik} \Rightarrow (8 + 8\sqrt{3}i)^{1/4} = 2e^{\pi i(1+6k)/12}$, $k = 0, 1, 2, 3$.
 Hence, the fourth roots of $8 + 8\sqrt{3}i$ are $2e^{\pi i/12}$, $2e^{7\pi i/12}$, $2e^{13\pi i/12}$, and $2e^{19\pi i/12}$.

8. Find the fourth roots of $-i$ and express them in the form $a + bi$, $a, b \in \mathbb{R}$. In order to evaluate the trigonometric functions, employ the identities $\cos^2(\theta) = \frac{1}{2}[1 + \cos(2\theta)]$ and $\sin^2(\theta) = \frac{1}{2}[1 - \cos(2\theta)]$.

Solution:

$-i = e^{-\pi i/2 + 2\pi i k} \Rightarrow (-i)^{1/4} = e^{\pi i(-1+4k)/8}$, $k = 0, 1, 2, 3$. Hence, the fourth roots of $-i$ are $e^{-\pi i/8} = \cos(\frac{\pi}{8}) - i \sin(\frac{\pi}{8})$, $e^{3\pi i/8} = \cos(\frac{3\pi}{8}) + i \sin(\frac{3\pi}{8})$, $e^{7\pi i/8} = \cos(\frac{7\pi}{8}) + i \sin(\frac{7\pi}{8})$, and $e^{11\pi i/8} = \cos(\frac{11\pi}{8}) + i \sin(\frac{11\pi}{8})$.

Let $\alpha = \cos(\frac{\pi}{8}) = \sqrt{\frac{1}{2}[1 + \cos(\frac{\pi}{4})]} = \sqrt{\frac{1}{2}(1 + \frac{1}{\sqrt{2}})}$ and

$\beta = \sin(\frac{\pi}{8}) = \sqrt{\frac{1}{2}[1 - \cos(\frac{\pi}{4})]} = \sqrt{\frac{1}{2}(1 - \frac{1}{\sqrt{2}})}$. Then the first root is $\alpha - \beta i$ and, similarly, the second, third and fourth are $\beta + \alpha i$, $-\alpha + \beta i$ and $-\alpha - \beta i$, respectively.

9. Find the square roots of $4 + i$ and express them in the form $a + bi$, $a, b \in \mathbb{R}$. Note that the values of $\cos(\frac{\theta}{2})$ and $\sin(\frac{\theta}{2})$ can be determined from the values of $\cos(\theta)$ and $\sin(\theta)$ without explicit knowledge of the value of θ .

Solution:

$4 + i = \sqrt{17}(\frac{4}{\sqrt{17}} + \frac{1}{\sqrt{17}}i) = \sqrt{17}e^{i\theta + 2\pi i k}$, where $\cos(\theta) = \frac{4}{\sqrt{17}}$ and $\sin(\theta) = \frac{1}{\sqrt{17}}$. Since $\cos(\theta) > 0$ and $\sin(\theta) > 0$, $\theta \in (0, \frac{\pi}{2})$. Then, $(4 + i)^{1/2} = \sqrt{\sqrt{17}}e^{i\theta/2 + \pi i k}$, $k = 0, 1$.

Hence, the square roots of $4 + i$ are $\sqrt{\sqrt{17}}e^{i\theta/2}$ and $\sqrt{\sqrt{17}}e^{i\theta/2 + \pi i} = -\sqrt{\sqrt{17}}e^{i\theta/2}$. Let

$\alpha = \cos(\frac{\theta}{2}) = \sqrt{\frac{1}{2}[1 + \cos(\theta)]} = \sqrt{\frac{1}{2}(1 + \frac{4}{\sqrt{17}})}$ and

$\beta = \sin(\frac{\theta}{2}) = \sqrt{\frac{1}{2}[1 - \cos(\theta)]} = \sqrt{\frac{1}{2}(1 - \frac{4}{\sqrt{17}})}$. Then the square roots are

$$\begin{aligned} \pm \sqrt{\sqrt{17}}(\alpha + i\beta) &= \pm \sqrt{\sqrt{17}} \left(\sqrt{\frac{1}{2}(1 + \frac{4}{\sqrt{17}})} + \sqrt{\frac{1}{2}(1 - \frac{4}{\sqrt{17}})}i \right) \\ &= \pm \left(\sqrt{\frac{1}{2}(\sqrt{17} + 4)} + \sqrt{\frac{1}{2}(\sqrt{17} - 4)}i \right). \end{aligned}$$

10. Find the square roots of $2 - i$ and express them in the form $a + bi$, $a, b \in \mathbb{R}$.

Solution:

$2 - i = \sqrt{5}(\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}}i) = \sqrt{5}e^{i\theta + 2\pi i k}$, where $\cos(\theta) = \frac{2}{\sqrt{5}}$ and $\sin(\theta) = -\frac{1}{\sqrt{5}}$. Since $\cos(\theta) > 0$ and $\sin(\theta) < 0$, $\theta \in (-\frac{\pi}{2}, 0) \Rightarrow \frac{\theta}{2} \in (-\frac{\pi}{4}, 0) \Rightarrow \cos(\frac{\theta}{2}) > 0$ and $\sin(\frac{\theta}{2}) < 0$.

Then $(2 - i)^{1/2} = \sqrt{\sqrt{5}}e^{i\theta/2 + \pi i k}$, $k = 0, 1$. Hence, the square roots of $2 - i$ are $\sqrt{\sqrt{5}}e^{i\theta/2}$

and $\sqrt{\sqrt{5}}e^{i\theta/2 + \pi i} = -\sqrt{\sqrt{5}}e^{i\theta/2}$. Let $\alpha = \cos(\frac{\theta}{2}) = \sqrt{\frac{1}{2}[1 + \cos(\theta)]} = \sqrt{\frac{1}{2}(1 + \frac{2}{\sqrt{5}})}$ and

$\beta = \sin(\frac{\theta}{2}) = -\sqrt{\frac{1}{2}[1 - \cos(\theta)]} = -\sqrt{\frac{1}{2}(1 - \frac{2}{\sqrt{5}})}$. Then the square roots are

$$\begin{aligned} \pm \sqrt{\sqrt{5}}(\alpha + i\beta) &= \pm \sqrt{\sqrt{5}} \left(\sqrt{\frac{1}{2}(1 + \frac{2}{\sqrt{5}})} - \sqrt{\frac{1}{2}(1 - \frac{2}{\sqrt{5}})}i \right) \\ &= \pm \left(\sqrt{\frac{1}{2}(\sqrt{5} + 2)} - \sqrt{\frac{1}{2}(\sqrt{5} - 2)}i \right). \end{aligned}$$

11. Find the cube roots of $-\frac{27}{\sqrt{2}} + \frac{27}{\sqrt{2}}i$ and express them in the form $a + bi$, $a, b \in \mathbb{R}$. In

order to evaluate the trigonometric functions, employ the identities

$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.

Solution:

$-\frac{27}{\sqrt{2}} + \frac{27}{\sqrt{2}}i = 27(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = 27e^{3\pi i/4 + 2\pi i k} \Rightarrow (-\frac{27}{\sqrt{2}} + \frac{27}{\sqrt{2}}i)^{1/3} = 3e^{\pi i(3+8k)/12}$, $k = 0, 1, 2$. Hence, the cube roots are $3e^{\pi i/4} = 3(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)$, $3e^{11\pi/12}$ and $3e^{19\pi/12}$.

$\cos(\frac{11\pi}{12}) = \cos(\frac{3\pi}{12} + \frac{8\pi}{12}) = \cos(\frac{\pi}{4})\cos(\frac{2\pi}{3}) - \sin(\frac{\pi}{4})\sin(\frac{2\pi}{3}) = -\frac{1}{2\sqrt{2}}(1 + \sqrt{3})$ and
 $\sin(\frac{11\pi}{12}) = \sin(\frac{\pi}{4})\cos(\frac{2\pi}{3}) + \cos(\frac{\pi}{4})\sin(\frac{2\pi}{3}) = \frac{1}{2\sqrt{2}}(\sqrt{3} - 1)$. Thus, the second cube root is $\frac{3}{2\sqrt{2}}[-(\sqrt{3} + 1) + (\sqrt{3} - 1)i]$. Similarly, employing $\frac{19\pi}{12} = \frac{11\pi}{12} + \frac{8\pi}{12}$, the third cube root is $\frac{3}{2\sqrt{2}}[(\sqrt{3} - 1) - (\sqrt{3} + 1)i]$.

12. (a) Show that $|e^{i\theta}| = 1$ for any $\theta \in \mathbb{R}$.

Solution:

$$|e^{i\theta}| = |\cos(\theta) + i\sin(\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1.$$

- (b) Show that if $z = re^{i\theta}$ is a polar representation, then $\bar{z} = re^{-i\theta}$.

Solution:

$$z = re^{i\theta} = r[\cos(\theta) + i\sin(\theta)] \Rightarrow \bar{z} = r[\cos(\theta) - i\sin(\theta)] = r[\cos(-\theta) + i\sin(-\theta)] = re^{-i\theta}.$$

- (c) Show that $\overline{e^z} = e^{\bar{z}}$ for any $z \in \mathbb{C}$.

Solution:

$$z = x + iy, x, y \in \mathbb{R} \Rightarrow e^z = e^{x+iy} = e^x e^{iy} \Rightarrow \overline{e^z} = e^x e^{-iy} = e^{x-iy} = e^{\bar{z}}.$$

13. Employ the definitions $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$ and $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$ in order to show that, for all $z, w \in \mathbb{C}$,

- (a) $\cos^2(z) + \sin^2(z) = 1$

Solution:

$$\begin{aligned} \cos^2(z) + \sin^2(z) &= \frac{1}{4}(e^{iz} + e^{-iz})^2 - \frac{1}{4}(e^{iz} - e^{-iz})^2 \\ &= \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) - \frac{1}{4}(e^{2iz} - 2 + e^{-2iz}) = 1. \end{aligned}$$

- (b) $\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$

Solution:

$$\begin{aligned} \sin(z)\cos(w) + \cos(z)\sin(w) &= \frac{1}{4i}(e^{iz} - e^{-iz})(e^{iw} + e^{-iw}) + \frac{1}{4i}(e^{iz} + e^{-iz})(e^{iw} - e^{-iw}) \\ &= \frac{1}{2i}[e^{i(z+w)} - e^{-i(z+w)}] = \sin(z+w). \end{aligned}$$

- (c) $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$

Solution:

$$\begin{aligned} \cos(z)\cos(w) - \sin(z)\sin(w) &= \frac{1}{4}(e^{iz} + e^{-iz})(e^{iw} + e^{-iw}) - \frac{1}{4}(e^{iz} - e^{-iz})(e^{iw} - e^{-iw}) \\ &= \frac{1}{2}[e^{i(z+w)} + e^{-i(z+w)}] = \cos(z+w). \end{aligned}$$

14. Employ the results of Problem 13 in order to show that, for all $z \in \mathbb{C}$,

- (a) $\sin(2z) = 2\sin(z)\cos(z)$

Solution:

Let $w = z$ in Problem 13(b).

- (b) $\cos(2z) = \cos^2(z) - \sin^2(z) = 2\cos^2(z) - 1 = 1 - 2\sin^2(z)$

Solution:

Let $w = z$ in Problem 13(c) and employ the result of Problem 13(a).

- (c) $\sin^2(z) = \frac{1 - \cos(2z)}{2}$

Solution:

Solve $\cos(2z) = 1 - 2\sin^2(z)$ in part (b) for $\sin^2(z)$.

- (d) $\cos^2(z) = \frac{1 + \cos(2z)}{2}$

Solution:

Solve $\cos(2z) = 2\cos^2(z) - 1$ in part (b) for $\cos^2(z)$.