McGill University April 2013

Faculty of Science Final examination

Differential geometry

Math 577

Friday, April 19th, 2013

Time: 2pm-5pm

Examiner: Prof. J. Hurtubise Associate Examiner: Prof. J. Walcher

INSTRUCTIONS

1. The questions have to be answered in the exam booklets provided.
2. This is a closed book exam. No crib sheet is allowed.
3. Calculators are not permitted.
4. Use of a regular dictionary is not permitted.
5. Use of a translation dictionary is permitted.

This exam comprises the cover page and two pages of questions, numbered 1 to 6.

1. (10 points) Consider the distribution $\mathcal{D}$ on $\mathbb{R}^3$ generated by

$$\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial y}$$

Is there, in a neighbourhood of the origin, a two-dimensional submanifold X of $\mathbb{R}^3$ such that the inclusion of X in $\mathbb{R}^3$ maps TX to $\mathcal{D}$? Justify your answer.

Sol'n: *If one takes the Lie bracket of the two vector fields, one finds* $-\frac{\partial}{\partial z}$*, which does not lie in the distribution, which then fails to be involutive, and so cannot be the tangent bundle of a submanifold.*

2. (30 points) One has $H^i(S^1 \times S^1) = \mathbb{R}$ for $i = 0, 2$, and $H^1(S^1 \times S^1) = \mathbb{R}^2$. Compute from the Mayer Vietoris sequence the deRham cohomology of $S^1 \times S^1 - \{\text{point}\}$, $S^1 \times S^1 - \{2 \text{ points}\}$, and then using this or otherwise, the deRham cohomology of the closed surface of genus g. You may assume standard results about the zero-th and top dimensional cohomology of compact and non-compact manifolds.

Sol'n:*If one covers the torus* $S^1 \times S^1$ *by a disk* D *and by* $V = S^1 \times S^1 - \{\text{point}\}$*, intersecting in a punctured disk* D^**, one has from Mayer-Vietoris that*

$$\begin{aligned} 0 \to &H^0(S^1 \times S^1) \to H^0(D) \oplus H^0(V) \to H^0(D^*) \\ \to &H^1(S^1 \times S^1) \to H^1(D) \oplus H^1(V) \to H^1(D^*) \\ \to &H^2(S^1 \times S^1) \to H^2(D) \oplus H^2(V) \to H^2(D^*) \to 0 \end{aligned}$$

The first line splits off as $0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to 0$*, as the components are all connected; as the disk is contractible to a point, and the punctured disk to a circle, and the punctured disk is not compact, so that* $H^2(V) = 0$*, one then has*

$$0 \to \mathbb{R}^2 \to H^1(V) \to \mathbb{R} \to \mathbb{R} \to 0,$$

giving $H^1(V) = \mathbb{R}^2$.

Now write V *as the union of the twice-punctured torus* W *and the disk* D*, intersecting on a punctured disk* W*; using Mayer-Vietoris again,*

$$\begin{aligned} 0 \to &H^0(V) \to H^0(D) \oplus H^0(W) \to H^0(D^*) \\ \to &H^1(V) \to H^1(D) \oplus H^1(W) \to H^1(D^*) \\ \to &H^2(V) \to H^2(D) \oplus H^2(W) \to H^2(D^*) \to 0 \end{aligned}$$

Again, as W *is connected,* $H^0(W) = \mathbb{R}$*; the sequence then gives you*

$$0 \to \mathbb{R}^2 \to H^1(W) \to \mathbb{R} \to 0 \to H^2(W) \to 0$$

giving $H^1(W) = \mathbb{R}^3, H^2(W) = 0$.

If one assumes inductively that $H^0(M) = 0, H^1(M) = \mathbb{R}^{2g}, H^2(M) = \mathbb{R}$ *for the surface of genus* g*, we then want to compute for* N *of genus* $g+1$*; removing a point from the surface* M *(call the result* M^**) computing as for* V *above gives* $H^0(M^*) = \mathbb{R}, H^1(M^*) = \mathbb{R}^{2g}, H^2(M^*) = 0$*; but then* *one can write N as the union of M^* and V, intersecting along an annulus, which topologically is D^*. One then has*

$$\begin{aligned} 0 \to &H^0(N) \to H^0(M^*) \oplus H^0(V) \to H^0(D^*) \\ \to &H^1(N) \to H^1(M^*) \oplus H^1(V) \to H^1(D^*) \\ \to &H^2(N) \to H^2(M^*) \oplus H^2(V) \to H^2(D^*) \to 0 \end{aligned}$$

Again, the first line splits off, giving $H^0(N) = \mathbb{R}$; the second and third lines become

$$\begin{aligned} 0 \to &H^1(N) \to \mathbb{R}^{2g} \oplus \mathbb{R}^2 \to \mathbb{R} \\ \to &H^2(N) \to 0 \end{aligned}$$

since $H^2(N) = \mathbb{R}$, then $H^1(N) = \mathbb{R}^{2g+2}$.

3. (20 points) a) Let M_n be the space of $n \times n$ real matrices, and Sym_n the linear subspace of symmetric matrices. Show that the map from M_n to Sym_n given by $A \mapsto AA^T$ is submersive on the set of A's mapping to the identity. The set of A's with $AA^T = \mathbf{I}$ forms the Lie group $O(n)$. Compute its Lie algebra.

b) The Lorenz group is the subgroup of linear transformations M of $\mathbb{R}^4$ which preserve the metric $P = \text{diag}(-1, 1, 1, 1)$, in the sense that $MPM^T = P$. Compute its Lie algebra.

Sol'n: *The differential of the map at A is $a \mapsto aA^T + Aa^T$; to see that it is submersive, we need to see that this is surjective onto the symmetric matrices when $AA^T = \mathbf{I}$, i.e. when $A^T = A^{-1}$. We note that the map $a \mapsto aA^T$ is surjective from M_n to M_n, as its inverse is $a \mapsto aA$; as the symmetrisation map $B \mapsto B + B^T$ is surjective also, the composition $a \mapsto aA^T + Aa^T$ is surjective.*

At $A = \mathbf{I}$, the differential of the map is $a \mapsto a + a^T$; the Lie algebra is the kernel of this , i.e. the skew symmetric matrices. This can be computed by $(\mathbf{I} + \epsilon a)(\mathbf{I} + \epsilon a)^T = \mathbf{I}$, and setting $\epsilon^2 = 0$.

For the Lorentz group, one has in a similar fashion $(\mathbf{I} + \epsilon a)P(\mathbf{I} + \epsilon a)^T = \mathbf{I}$, giving for the order ϵ term $aP + Pa^T = 0$.

4. (10 points) (Hamiltonian mechanics). Let M be an even dimensional manifold, and let ω be a symplectic form on M, that is a non-degenerate closed two-form. Let $I : T^*M \to TM$ be the isomorphism induced by the two-form. For any function H on M, set $X_H = I(dH)$. Show that $\mathcal{L}_{X_H}(H) = 0$, $\mathcal{L}_{X_H}(\omega) = 0$.

Sol'n:*The defining relation for I is that $\omega(X_H, V) = dH(V)$ for any V. One has $\mathcal{L}_{X_H}(H) = X_H(H) = dH(X_H) = \omega(X_H, X_H) = 0$, as ω is skew-symmetric. For $\mathcal{L}_{X_H}(\omega)$ one uses the Cartan relation $\mathcal{L}_{X_H} = i_{X_H} \circ d + d \circ i_{X_H}$, so that on ω, which is closed, $\mathcal{L}_{X_H}(\omega) = d \circ i_{X_H}(\omega)$. However, by definition, $i_{X_H}(\omega)(V) = \omega(X_H, V) = dH(V)$, so that $d \circ i_{X_H}(\omega) = d \circ dH = 0$.*

5. (10 points) Consider the map F of the torus $S^1 \times S^1 \to \mathbb{R}^3$, given by

$$F(u, v) = ((R + r\cos(u))\cos(v), (R + r\cos(u))\sin(v), r\sin(u)),$$

where $r < R$, and u, v are angular coordinates. Compute the vector fields $F_*(\frac{\partial}{\partial u})$, $F_*(\frac{\partial}{\partial v})$ as well as their Lie bracket. Compute the Lie derivative $\mathcal{L}_{\frac{\partial}{\partial u}}(F^*(dx))$.

Sol'n:*One has for the components in the basis* $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$,

$$\frac{\partial x}{\partial u} = -r\sin(u)\cos(v),\ \frac{\partial y}{\partial u} = -r\sin(u)\sin(v),\ \frac{\partial z}{\partial u} = r\cos(u)),$$
$$\frac{\partial x}{\partial v} = -(R + r\cos(u))\sin(v),\ \frac{\partial y}{\partial v} = (R + r\cos(u))\cos(v),\ \frac{\partial z}{\partial v} = 0,$$

For the Lie bracket, it must be zero, as this is the image by F of vector fields which commute on the torus. For the Lie derivative, the Cartan relation gives $\mathcal{L}_{\frac{\partial}{\partial u}}(F^*(dx)) = d(\frac{\partial x}{\partial u}) = d(-r\sin(u)\cos(v)) = -r\cos(u)\cos(v)du + r\sin(u)\sin(v)dv$.

6. (20 points) Consider the two-sphere S^2 along with its natural inclusion into $\mathbb{R}^3$ as the set of points of distance one from the origin. Let x, y be the coordinates on the sphere corresponding to projection onto the disk, so that (x, y) corresponds to the point $(x, y, \sqrt{1 - x^2 - y^2})$. The tangent bundle $T\mathbb{R}^3$ restricts to S^2, giving a trivial rank 3 bundle $S^2 \times \mathbb{R}^3$, equipped with the trivial connection d. Writing the sections of this bundle as functions $S^2 \to \mathbb{R}^3$, this connection is simply $(f_1, f_2, f_3) \mapsto (df_1, df_2, df_3)$.

Along S^2 there is a natural subline bundle L of $T\mathbb{R}^3|_{S^2}$, given by the radial vectors: at a point p of S^2, the line L_p is simply generated the line in $\mathbb{R}^3 = T_p\mathbb{R}^3$ generated by p. This subbundle L has a natural length one section s, given in the projected coordinates x, y of the two-sphere onto the disk by $s(x, y) = (x, y, \sqrt{1 - x^2 - y^2}))$. There is a natural projection π of $T\mathbb{R}^3|_{S^2} = S^2 \times \mathbb{R}^3$ onto L, given by

$$\pi(x, y, a', b', c') = (x, y, < s(x, y), (a', b', c') > \cdot s(x, y)),$$

where $<,>$ is the scalar product. One can define a connection ∇ on L by $\nabla(\hat{s}) = \pi \circ d(\hat{s})$.

a) Show that this is indeed a connection on L, i.e, that the defining relations of a connection as a map from $\Gamma(L) \to \Gamma(L \otimes T^*S^2)$ are satisfied.

b) For the canonical section s compute the derivative ds, i.e. evaluate the connection on s thought of as a section of the trivial bundle $T\mathbb{R}^3$, and then compute $\nabla(s)$. Compute the curvature of ∇, acting on s.

c) (Extra marks) The bundle map $\mathbf{I} - \pi$ defines a projection not onto L, but onto its orthogonal complement, the tangent bundle of S^2, and one has a similar connection $\nabla' = (\mathbf{I} - \pi) \circ d$ on sections of $TS^2 \subset T\mathbb{R}^3$. Show that this is the Levi-Civita connection of TS^2.

Sol'n *a) To see that it is a connection, one needs the two defining relations of a connection, that is linearity and the Leibniz rule. The first is obvious, and the second follows because the trivial connection d satisfies the Leibniz rule, and the projection π is linear.*

b) The computation of ds is straightforward, and we note that as the length of s is constant, its derivative is orthogonal to it, so that the projection back to L is zero; one has $\nabla(s) = 0$. It then follows that the curvature is zero, too.

c) By the same argument as in a), if $\rho = \mathbf{I} - \pi$ is the projection from the trivial rank 3 bundle back to the tangent bundle of S^2, one has that $\nabla = \rho \circ d$ is a connection. For it to be the Levi-Civita connection, it must then preserve the metric, and be torsion free. For the metric part, one wants

$$d < s, t > = < \nabla s, t > + < s, \nabla t >$$

One certainly has that as sections of the trivial rank 3 bundle, that $d < s,t > = < ds, t > + < s, dt >$*; but then if* s *lies in the tangent space to* S^2*, then* $< s.s' > = < s, \rho s' >$ *for any* s'*; and the same holds for* t*.*

For the torsion free property, one wants that on 1-forms ω*, that skew-symmetrising* $\nabla\omega$ *simply gives* $d\omega$*. On forms the projection is the same as pull back; and taking the component-wise derivatives in* $\mathbb{R}^3$*, then skew symmetrising is precisely the way one computes the exterior derivative of a one-form.*