School of Mathematics and Statistics
Carleton University
Math. 1005A, Winter 2011

Mock TEST 6

Any non-programmable calculator permitted, 1 blank sheet permitted for roughs

Print Name : Solutions

Student Number:

Tutorial Section (A1, A4, ...):

PART I: Multiple Choice Questions

(Choose and CIRCLE only ONE answer - No part marks here.)

1. **[2 marks]** Which of the following statements is true about the Fourier series of the function f defined on $(-\pi, \pi)$ by $f(x) = -1, -\pi < x < 0$ and $f(x) = 1, 0 < x < \pi$?
   (a) $a_n = 0$ for all $n \geq 1$, (b) $b_2 = 0$, (c) $a_0 \neq 0$, (d) $b_n = 0$ for all $n \geq 1$.

2. **[2 marks]** Find the Fourier cosine series of the function f defined on $(0, \pi)$ by $f(x) = 1, 0 < x < 1$ and $f(x) = 0, 1 < x < \pi$.
   (a) $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n}{n}\cos nx$, (b) $\frac{1}{\pi} + \sum_{n=1}^{\infty}\frac{\sin n}{n}\cos nx$ (c) $\frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n}{n^2}\cos nx$ (d) $\frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n}{n}\cos nx$.

3. **[2 marks]** The first term term, b_1, of the Fourier sine series of the function $f(x) = x^2$ for $0 < x < \pi$ is given by:
   (a) 0, (b) $\frac{8}{\pi}\sin x$, (c) $2\pi - \frac{8}{\pi}\sin x$, (d) $2\pi + \frac{8}{\pi}\sin x$.

4. **[2 marks]** Find the first term, a_0, of the Fourier cosine series of the function $f(x) = \frac{\pi}{4} - \frac{x}{2}$ where $0 < x < \pi$.
   (a) $a_0 = \frac{2}{\pi}$, (b) $a_0 = 0$, (c) $a_0 = 1$, (d) $a_0 = \frac{\pi}{2}$.

5. **[2 marks]** Among the following pairs of functions (f, g), (g, h) and (f, h) where

$$f(x) = \cos x, \quad g(x) = \sin^2 x, \quad h(x) = x^2,$$

   which pairs represent functions that are orthogonal to each other on the interval $(0, \pi)$?
   (a) (f, h) but not (f, g), (b) (f, g) but not (f, h) or (g, h), (c) (g, h) but not (f, g) or (f, h), (d) (f, h) but not (g, h).

PART II: Show all work here and give details.

No additional pages will be accepted

6. [10 marks] Let $f(x) = \left\{ \begin{array}{ll} 1, & -1 \leq x \leq 0 \\ x+1, & 0 < x \leq 1 \end{array} \right\}$. Find the Fourier series of f in the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right]$$

where $n \geq 1$. $L = 1$.

$$\therefore a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos n\pi x\,dx = \left(\int_{-1}^{0} + \int_{0}^{1}\right) f(x)\cos n\pi x\,dx$$

$$= \int_{-1}^{0}\cos n\pi x\,dx + \int_{0}^{1}(x+1)\cos n\pi x\,dx$$

$x+1 \ (+)$ $\cos n\pi x$
$1 \ (-)$ $\frac{\sin n\pi x}{n\pi}$
0 $-\frac{\cos n\pi x}{n^2\pi^2}$

$$= \left.\frac{\sin n\pi x}{n\pi}\right|_{x=-1}^{x=0} + \left.\left[(x+1)\frac{\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]\right|_{x=0}^{x=1}$$

$\Rightarrow$

$$\therefore \boxed{a_n = \frac{(-1)^n - 1}{n^2\pi^2}}.$$

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x)\sin n\pi x\,dx \\ &= \int_{-1}^{0} \sin n\pi x\,dx + \int_0^1 (x+1)\sin n\pi x\,dx \\ &= \frac{(-1)^n - 1}{n\pi} + \frac{-2n\pi(-1)^n + n\pi}{n^2\pi^2} \\ &= \frac{(-1)^n - 1 + 1 - 2(-1)^n}{n\pi} = \boxed{\frac{(-1)^{n+1}}{n\pi}}, \quad n \geq 1 \end{aligned}$$

7. [10 marks] Let $f(x) = x^2$ for $0 \leq x \leq 1$. Find the a_n in the half-range cosine series of f (which is of the form $\sum_{n=1}^{\infty} a_n \cos(n\pi x)$)

$$a_n = \frac{2}{L}\int_0^L f(x)\cos n\pi x, \quad L = 1.$$

$$= 2\int_0^1 x^2 \cos n\pi x\,dx.$$

$$\begin{array}{c|c} x^2 \ (+) & \cos n\pi x \\ 2x \ (-) & \frac{\sin n\pi x}{n\pi} \\ 2 \ (+) & -\frac{\cos n\pi x}{n^2\pi^2} \\ 0 & -\frac{\sin n\pi x}{n^3\pi^3} \end{array}$$

$$= 2\left[x^2\frac{\sin n\pi x}{n\pi} + 2\frac{x\cos n\pi x}{n^2\pi^2} - \frac{2\sin n\pi x}{n^3\pi^3}\right]\Bigg|_{x=0}^{1}$$

$$= \frac{2^2(-1)^n \pi n}{n^3\pi^3} = \boxed{\frac{2^2(-1)^n}{n^2\pi^2}}, \quad \forall n \geq 1.$$

$$\therefore x^2 = 4\sum_{n\geq 1} \frac{(-1)^n}{n^2\pi^2}\cos n\pi x + \frac{2/3}{2} \to 1/3$$

$$\left(\therefore a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^1 x^2\,dx = 2\cdot\frac{1}{3} = \boxed{\frac{2}{3}}\right)$$

a_0.