COMP 233 Probability and Statistics for Computer Science

Fall 2016, Assignment 1 Due: September 30, 2016

Question 1:

Four bits are transmitted over a digital communications channel. Each bit is received either distorted or not distorted. Let A_i denote the event that the ith bit is distorted, $i = 1,\ldots,4$.

(a) Describe the sample space for this experiment.

(b) Are the Ai's mutually exclusive?

Describe the outcomes in each of the following events:

(c) A_1

(d) A_1^c

(e) $A_1 \cap A_2 \cap A_3 \cap A_4$

(f) $(A_1 \cap A_2) \cup (A_3 \cap A_4)$

You may either explicitly list the outcomes in the event, or give a concise description of all the outcomes in the event.

Solution.

Let d *and* o *denote a distorted bit and one that is not distorted (*o *denotes okay), respectively.*

a) $$S = \left\{\begin{matrix} dddd, dodd, oddd, oodd, \\ dddo, dodo, oddo, oodo, \\ ddod, dood, odod, oood, \\ ddoo, dooo, odoo, oooo \end{matrix}\right\}$$

b) No, for example $A_1 \cap A_2 = \{dddd, dddo, ddod, ddoo\}$

c) $$A_1 = \left\{\begin{matrix} dddd, dodd, \\ dddo, dodo \\ ddod, dood \\ ddoo, dooo \end{matrix}\right\}$$

d) $$A_1^c = \left\{\begin{matrix} oddd, oodd, \\ oddo, oodo, \\ odod, oood, \\ odoo, oooo \end{matrix}\right\}$$

e) $A_1 \cap A_2 \cap A_3 \cap A_4 = \{dddd\}$

f) $(A_1 \cap A_2) \cup (A_3 \cap A_4) = \{dddd, dodd, dddo, oddd, ddod, oodd, ddoo\}$

Question2:

(a) Prove that $P(EF^c) = P(E) - P(EF)$
(b) Prove that $P(E^cF^c) = 1 - P(E) - P(F) + P(EF)$
(c) Show that the probability that exactly one of the events E or F occurs is equal to $P(E)+P(F)-2P(EF)$

Solution.

(a) Write $E = EF \cup EF^c$ and apply Axiom 3.
(b) $P(E^cF^c) = P(E^c) - P(E^cF)$ from part (a)
$= 1 - P(E) - [P(F) - P(EF)]$
(c) $P(EF^c \cup E^cF) = P(EF^c) + P(E^cF)$
$= P(E) - P(EF) + P(F) - P(EF)$ from (a)

Question 3:

A batch of 140 semiconductor chips is inspected by choosing a sample of five chips. Assume 10 of the chips do not conform to customer requirements.

(a) How many different samples are possible?

(b) How many samples of five contain exactly one non-conforming chip?

(c) How many samples of five contain at least one non-conforming chip?

Solution.

a) From equation 2-4, the number of samples of size five is $\binom{140}{5} = \frac{140!}{5!135!} = 416965528$

b) There are 10 ways of selecting one nonconforming chip and there are $\binom{130}{4} = \frac{130!}{4!126!} = 11358880$ ways of selecting four conforming chips. Therefore, the number of samples that contain exactly one nonconforming chip is $10 \times \binom{130}{4} = 113588800$.

c) The number of samples that contain at least one nonconforming chip is the total number of samples $\binom{140}{5}$ minus the number of samples that contain no nonconforming chips $\binom{130}{5}$. That is $\binom{140}{5} - \binom{130}{5} = \frac{140!}{5!135!} - \frac{130!}{5!125!} = 130721752$

Question 4:

In a chemical plant, 24 holding tanks are used for final product storage. Four tanks are selected at random and without replacement. Suppose that six of the tanks contain material in which the viscosity exceeds the customer requirements.

(a) What is the probability that exactly one tank in the sample contains high-viscosity material?

(b) What is the probability that at least one tank in the sample contains high-viscosity material?

(c) In addition to the six tanks with high viscosity levels, four different tanks contain material with high impurities. What is the probability that exactly one tank in the sample contains high-viscosity material and exactly one tank in the sample contains material with high impurities?

Solution.

a) The total number of samples possible is $\binom{24}{4} = \frac{24!}{4!20!} = 10{,}626$. The number of samples in which exactly one tank has high viscosity is $\binom{6}{1}\binom{18}{3} = \frac{6!}{1!5!} \times \frac{18!}{3!15!} = 4896$. Therefore, the probability is $\frac{4896}{10626} = 0.461$

b) The number of samples that contain no tank with high viscosity is $\binom{18}{4} = \frac{18!}{4!14!} = 3060$. Therefore, the requested probability is $1 - \frac{3060}{10626} = 0.712$.

c) The number of samples that meet the requirements is $\binom{6}{1}\binom{4}{1}\binom{14}{2} = \frac{6!}{1!5!} \times \frac{4!}{1!3!} \times \frac{14!}{2!12!} = 2184$. Therefore, the probability is $\frac{2184}{10626} = 0.206$.

Question 5:

Suppose you have to select one project partner from a set of four classmates, who have different GPAs. Assume you do not know any student's GPA in advance but can get to know it after you have picked a student from that group

(a) Suppose you pick one of the four students at random and accept that student as your project partner. What is the probability that your partner is the one with the highest GPA?

(b) Suppose you decide to reject the first student and to then accept the next student if and only if that student has a higher GPA. Note that you MUST have a partner, so if the first three are rejected by you, then you have to accept the fourth student. What is the probability that your partner will be the one with the highest GPA.

Solution.

Let 1 … 4 denote the students in the group in increasing GPAs.

(a) 1/4, since the first student randomly picked is equally likely to be any of the 4 students.

(b) The sample space consists of 24 orderings as follows:

1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, 2341, 2413, 2431, 3124, 3142, 3214, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312, 4321

You will have the highest GPA partner if the students appear in any of the following orderings:

1423, 1432, 2143, 2413, 2431, 3124, 3142, 3214, 3241, 3412, 3421

Thus, with probability 11/24 you will accept the highest GPA.

Question 6:

Suppose that $f(x) = e^{-x}$ for $0 < x$. Determine the following:

(a) $P(1 < X)$

(b) $P(1 < X < 2.5)$

(c) $P(X = 3)$

(d) $P(X < 4)$

(e) $P(3 \le X)$

(f) Determine x such that $P(x < X) = 0.10$.

(g) Determine x such that $P(X \le x) = 0.10$.

Solution.

a) $P(1 < X) = \int_1^{\infty} e^{-x} dx = (-e^{-x})\Big|_1^{\infty} = e^{-1} = 0.3679$

b) $P(1 < X < 2.5) = \int_1^{2.5} e^{-x} dx = (-e^{-x})\Big|_1^{2.5} = e^{-1} - e^{-2.5} = 0.2858$

c) $P(X = 3) = \int_3^{3} e^{-x} dx = 0$

d) $P(X < 4) = \int_0^{4} e^{-x} dx = (-e^{-x})\Big|_0^{4} = 1 - e^{-4} = 0.9817$

e) $P(3 \le X) = \int_3^{\infty} e^{-x} dx = (-e^{-x})\Big|_3^{\infty} = e^{-3} = 0.0498$

f) $$P(x < X) = \int_x^\infty e^{-x} dx = (-e^{-x})\Big|_x^\infty = e^{-x} = 0.10$$

Then, x = – ln(0.10) = 2.3.

g) $$P(X \leq x) = \int_0^x e^{-x} dx = (-e^{-x})\Big|_0^x = 1 - e^{-x} = 0.10$$

Then, x = –ln(0.9) = 0.1054

Question 7:

The joint probability density function of random variables X and Y is given by

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right), \qquad 0 < x < 1, \qquad 0 < y < 2$$

(a) Verify that this is indeed a joint density function.
(b) Compute the density function of X.
(c) Find P{X > Y}.

Solution.

(a) Show that the multiple integral of the joint density equals 1:

$$\int_0^1 \int_0^2 f(x,y) dy\, dx = \frac{6}{7}\int_0^1 \left(\left(x^2 y + \frac{x}{2}\frac{y^2}{2}\right)\Big|_0^2\right) dx = \frac{6}{7}\int_0^1 (2x^2 + x) dx = \frac{6}{7}\left(\frac{2x^3}{3} + \frac{x^2}{2}\right)\Big|_0^1 = \frac{6}{7}\left(\frac{2}{3} + \frac{1}{2}\right) = \frac{6}{7}\left(\frac{7}{6}\right) = 1.$$

(b) $\int_0^2 f(x,y) dy = 12x^2/7 + 6x/7.$

(c) $\int_0^1 \int_0^x f(x,y) dy\, dx = \int_0^1 (6x^3/7 + 3x^3/14) dx = 15/56.$