University of Ottawa
MAT1330 Midterm Exam

October 5, 2015. Duration: 80 minutes. Instructor: Frithjof Lutscher.

Family Name: Solutions Version A

First Name: ____________________

DGD 1 DGD 2 DGD 3 DGD 4

Do **not** write your student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: you do not have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, pagers or any text storage or communication device **is not permitted.**
- Only the Faculty approved calculators (TI-30X, TI-34X, Casio FX-260X and Casio FX-300X) are allowed.
- Questions 1–5 are multiple choice questions. You MUST record your answers in the boxes at the top of page 2. Each of these questions is worth 2 points.
- Questions 6 and 7 are long-answer questions. The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Answer these questions in the space provided. Use the backs of pages for additional calculations if necessary.
- Where it is possible to check your work, do so.
- Please do not detach the pages.
- Good luck!

Student number:__________________, Total marks: __________ out of 30

Problem	1	2	3	4	5	6	7
Answers to 1–5	A	B	E	E	E	X	X
Marks							

Question 1. If $f(x) = 5x - 1$ and $g(x) = -3x + 2$ what is $(f \circ g)(1)$?

Answer: A: -6; B: 6; C: -4; D: 4; E: -10.

$(f\circ g)(x) = f(g(x)) = 5(-3x+2)-1 = -15x+10-1 = -15x+9$

$(f\circ g)(1) = -15(1)+9 = -15+9 = -6$

Question 2. What is the domain of the function

$$f(x) = \frac{x+1}{x^2-2}$$

Answer:
A: all numbers except 0; B: all numbers except $\pm\sqrt{2}$; C: all numbers greater than 2;
D: all numbers except ± 2; E: all numbers greater than $\sqrt{2}$

$x^2 - 2 \neq 0$

$x^2 \neq 2$

$x \neq \pm\sqrt{2}$

Question 3. Which of the following statements is true?
(i) If $a, b > 0$ then $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$. → False
(ii) $\arccos(x) = \frac{1}{\cos(x)}$ → False
(iii) If $a, b > 0$ then $\ln(a) + \ln(b) = \ln(ab)$. → True
(iv) $(a+b)^2 = a^2 + b^2$ → False
(v) $e^{x+y} = e^x + e^y$ → False

Answer: A: (i), (ii) and (iii); B: (ii) and (iv); C: (ii), (iii) and (v); D: (ii) and (iii) only; E: (iii) only.

Question 4. Find all solutions of the equation $|x^3 - 5| = 5$.

Answer: A: 0; B: $0, \pm\sqrt[3]{10}$; C: $0, \pm\sqrt{5}$; D: $0, \pm\sqrt[3]{5}$; E: $0, \sqrt[3]{10}$

Case ①

$x^3 - 5 = 5$

$x^3 = 10$

$x = \sqrt[3]{10}$

case ②:

$-(x^3-5) = 5$

$x^3 - 5 = -5$

$x^3 = 0$

$x = 0$

Question 5. Find all solutions of $\log(x+2) + \log(x+3) = \log(2)$.

Answer: A: -5; B: $1, -5$; C: $1, 5$; D: $-1, -5$; E: -1

$\log(x+2)(x+3) = \log(2)$

$(x+2)(x+3) = 2$

$x^2 + 5x + 6 = 2$

P:4 4×1

$x^2 + 5x + 4 = 0$ S:5

$(x+4)(x+1) = 0$

$\cancel{x = -4}$ → Not in the domain! & $x = -1$

Question 6. [13 points] The number of fish in a lake decreases by 30% each year since anglers take out more than fish reproduce. To prevent a collapse of the population, the municipality decides to restock 900 fish at the end of each year. The DTDS for the number of fish in the lake in year t, denoted by x_t, is given by $x_{t+1} = 0.7x_t + 900$.

(a) The updating function of the DTDS is $f(x) =$ $\boxed{0.7x + 900}$

(b) The general solution of the DTDS is $x_t =$ $\boxed{0.7^t(x_0 - 3000) + 3000}$

(c) The steady state of the DTDS is $x^* =$ $\boxed{3000}$

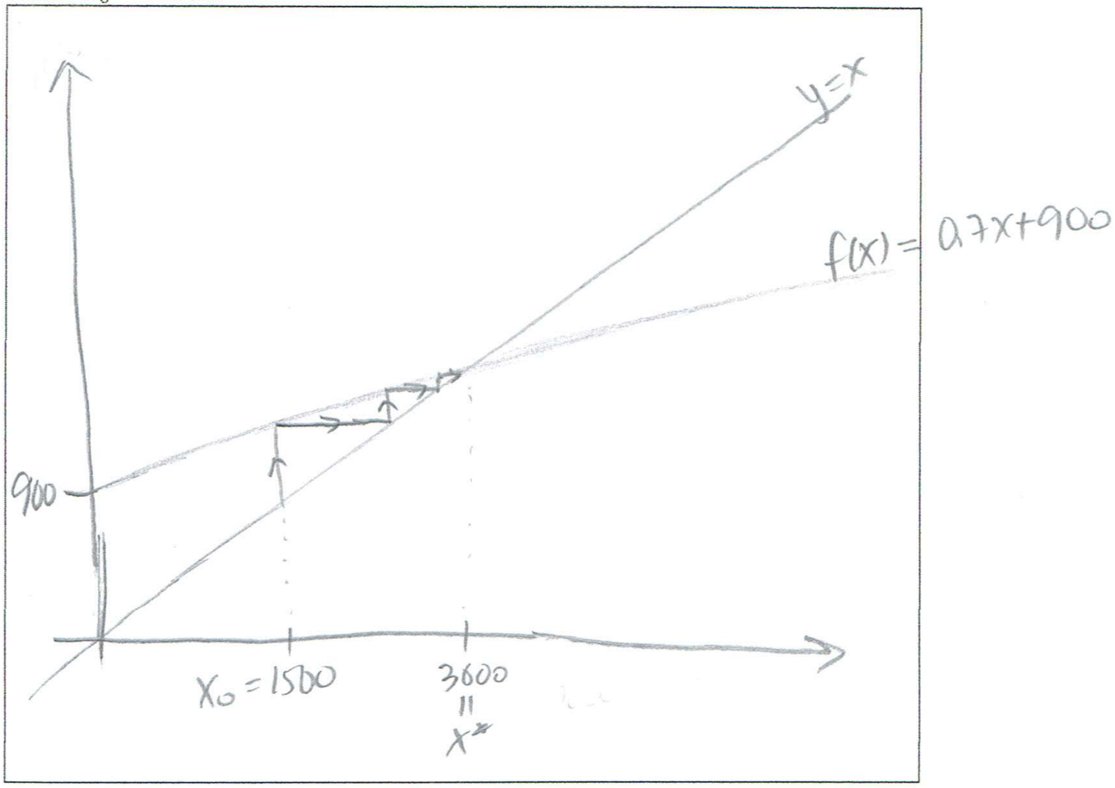
$x^* = 0.7x^* + 900$
$0.3x^* = 900$ $\Rightarrow x^* = 3000$

(d) Early one year, after a particularly cold and long winter, the municipality counts only $x_0 = 1500$ fish in the lake. Calculate x_1, x_2, x_3.

Answer $\boxed{x_1 = 1950;\ x_2 = 2265;\ x_3 = 2485}$

$x_0 = 1500$
$x_1 = 0.7(1500) + 900 = 1950$
$x_2 = 0.7(1950) + 900 = 2265$
$x_3 = 0.7(2265) + 900 = 2485$

(e) Draw the graph of the updating function and cobweb the solution of the DTDS starting from $x_0 = 1500$.



(f) Is the steady state stable? Justify your answer through **two** different ways of reasoning.

The steady state is Stable because:

① The solution in the cobweb shows that it is going towards the steady state and

② The value of r in $x_{t+1} = rx_t + c$ is $|r| = |0.7| = 0.7 < 1$, meaning it's stable.

(g) How long will it take for a population of $x_0 = 1500$ fish to grow up to 80% of the steady state value?

Answer: 3 years

We want our population X_t to reach 80% of 3000, so 2400.

We need to solve for t when $X_t = 2400$.

$$2400 = 0.7^t(1500-3000)+3000$$
$$-600 = 0.7^t(-1500)$$
$$0.4 = 0.7^t$$
$$\ln 0.4 = t \ln 0.7$$
$$t = \frac{\ln 0.4}{\ln 0.7} = 2.569$$

$\therefore$ It will take 3 years to reach 2400 fish.

(h) Bonus (2 points) The municipality wants to increase the steady state level of fish in the lake to 4000, but they cannot afford a larger restocking program. Therefore, they decide to limit fishing. What is the percentage of annual decrease (in percent) that they can allow to get the steady state to be 4000?

22.5%

We want $X^* = 4000$. → but we can't increase the "restock" $X_{t+1} = 0.7X_t + 900$

We want to find r in $X_{t+1} = rX_t + 900$ in order to have $X^* = 4000$.

$$X^* = rX^* + 900$$
$$4000 = r(4000) + 900$$
$$3100 = 4000r$$
$$0.775 = r$$

Thus, instead of having an annual decrease of 30% each year (this gives $X^* = 3000$), we need to limit this to an annual decrease of 22.5%.

Question ~~8~~ 7. [7 points] The plot below shows you the graph of an updating function $y = f(x)$ and the diagonal $y = x$.

(a) How many steady states does the DTDS $x_{t+1} = f(x_t)$ have and what are their approximate values? Mark them in the plot.

Answer: 3 steady states ; $x^* = 0.1, 0.35, 0.7$

(b) Start the cobwebbing process at the value $x_0 = 0.3$. What is the long-term behavior of the corresponding solution?

Answer: Over time, population equilibrates at $x^* = 0.1$

(c) Start the cobwebbing process at the value $x_0 = 0.5$. What is the long-term behavior of the corresponding solution?

Answer: Over time, population equilibrates at $x^* = 0.1$

(d) Based on your cobwebbing, indicate which of the steady states seem to be stable and which seem unstable.

Answer: $x^* = 0.1$, stable ; $x^* = 0.35$, unstable ; $x^* = 0.7$, unstable