COMP - 1805
Discrete Structures

§1 Logic and Proofs

If I am in the city of Ottawa, A
then I am in the province of Ontario. B

If I am Jean-Lou De Carufel, C
then my email address is jdecaruf@cg.scs.carleton.ca. D

What is the difference between these two sentences?

How can we analyze such things?

Def: A proposition is a declarative sentence that is either true or false (but not both).

We can combine propositions to get new propositions, as in the introductory examples. We do it by using logical operators : $\neg$ (negation)

$\wedge$ (conjunction)

$\vee$ (disjunction)

$\rightarrow$ (~~implication~~ conditional)

$\leftrightarrow$ (biconditional)

$\oplus$ (exclusive or)

We present the behaviour of these operators with truth tables.

Negation

p	$\neg p$
T	F
F	T

p : It rains.

$\neg p$: It does not rain

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p : It rains.

q : ~~2+3=4~~ I am a man.

$p \wedge q$: It rains and ~~2+3=4.~~ I am a man.

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	P	F

$p \vee q$: It rains or ~~2+3=4~~ I am a man.

Conditional

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p: It rains.

q: I wear my raincoat.

$p \to q$: If it rains,

then ~~2+3=4.~~

I wear my raincoat.

Biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

p: I am Jean-Lou De Carufel

q: My email address is jdecaruf.......

$p \leftrightarrow q$: ~~It rains if and only if 2+3=4.~~

I am Jean-Lou De Carufel if and only if my email address is jdecaruf.....

Exclusive or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

~~$p \oplus q$: It rains and 2+3=4~~

~~so it does not rains~~

2 special propositions:

TRUE : always true

FALSE : always false

Def: If the main column of the truth table of a proposition P only contains T, then P is a Tautology. If it only contains F, then P is a contradiction.

ex: $P \vee \neg P$ is a tautology

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

$P \wedge \neg P$ is a contradiction

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Def If $p \leftrightarrow q$ is a tautology, then we say that p and q are logically equivalent. We write $p \equiv q$. or $p \Leftrightarrow q$.

We can intuitively think of $p \equiv q$ as "p and q are two different ways of saying exactly the same thing".

ex: $p \to q \leftrightarrow \neg p \vee q$ is a tautology

p	q	$p \to q$	$\neg p \vee q$	$p \to q \leftrightarrow \neg p \vee q$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

So $p \to q \equiv \neg p \vee q$

So $p \to q$ and $\neg p \vee q$ are logically equivalent.

Whatever p and q are, from $p \to q$ you can always deduce $\neg p \vee q$ and vice-versa.

ex: $p \to q \leftrightarrow q \to p$ is not a tautology

p	q	$p \to q$	$q \to p$	$p \to q \leftrightarrow q \to p$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

$p \to q$ and $q \to p$ are not logically equivalent.
If $p \to q$ is true, it does not mean that $q \to p$ is.
Remember the introductory example...

p: I am in the city of Ottawa

q: I am in the province of Ontario

$p \to q$

$\neg p \vee q$

$\cancel{q \to p}$

$\neg q \to \neg p$

Is $p \to q \equiv \neg q \to \neg p$?

p	q	$p \to q$	$\neg q \to \neg p$	$p \to q \leftrightarrow \neg q \to \neg p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Yes !

Rules of Logic

$p \wedge \text{TRUE} \equiv p$
$p \vee \text{FALSE} \equiv p$ | Identity

$p \vee \text{TRUE} \equiv \text{TRUE}$
$p \wedge \text{FALSE} \equiv \text{FALSE}$ | Domination

$p \vee p \equiv p$
$p \wedge p \equiv p$ | Idempotent

$\neg(\neg p) \equiv p$ Double negation

$p \vee q \equiv q \vee p$
$p \wedge q \equiv q \wedge p$ | Commutative

$(p \vee q) \vee r \equiv p \vee (q \vee r)$
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | Associative

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | Distributive

$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ | De Morgan

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$p \lor (p \land q) \equiv p$ | Absorption
$p \land (p \lor q) \equiv p$

* $p \lor \neg p \equiv \text{TRUE}$ | Negation
* $p \land \neg p \equiv \text{FALSE}$

* $p \to q \equiv \neg p \lor q$ (1)
* $p \to q \equiv \neg q \to \neg p$ (2)

$p \lor q \equiv \neg p \to q$ (3)

$p \land q \equiv \neg(p \to \neg q)$ (4)

$\neg(p \to q) \equiv p \land \neg q$ (5)

$(p \to q) \land (p \to r) \equiv p \to (q \land r)$ (6)

$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$ (7)

$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$ (8)

$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$ (9)

We could do the truth table of all these laws. We would find that they are indeed all tautologies. Let's look at a few of them.

$p \wedge \text{TRUE} \equiv p$

~~$p \vee \text{TRUE}$~~

$p \wedge \text{FALSE} \equiv \text{FALSE}$

De Morgan

$\neg(p \to q) \equiv p \wedge \neg q$

Then we can use these laws to prove more laws.

ex: Prove that $\neg(A \to B) \to A$ is a tautology.

$\neg(A \to B) \to A$

$\equiv \neg(\neg(A \to B)) \vee A$ 1

$\equiv (A \to B) \vee A$ Double Negation Law

$\equiv (\neg A \vee B) \vee A$ 1

$\equiv (B \vee \neg A) \vee A$ Commutative Law

$\equiv B \vee (\neg A \vee A)$ Associative Law

$\equiv B \vee (A \vee \neg A)$ Commutative Law

$\equiv B \vee \text{TRUE}$ Negation Law

$\equiv \text{TRUE}$ Domination Law

What does it mean that

$\neg(A \rightarrow B) \rightarrow A \equiv \text{TRUE}$?

What does it mean that

$\neg(A \rightarrow B) \rightarrow A$ is a tautology?

ex: Prove that $\neg((A \wedge B) \rightarrow (A \rightarrow B))$ is a contradiction

$\neg((A \wedge B) \rightarrow (A \rightarrow B))$

$\equiv \neg(\neg(A \wedge B) \vee (A \rightarrow B))$ (1)

$\cancel{\equiv \neg((A \wedge B) \vee \neg(A}$

$\cancel{\equiv \neg(\neg(A \wedge B))}$

$\equiv \neg(\neg(A \wedge B) \vee (\neg A \vee B))$ (1)

$\equiv \neg(\neg(A \wedge B)) \wedge \neg(\neg A \vee B)$ De Morgan

$\equiv (A \wedge B) \wedge \neg(\neg A \vee B)$ Double Negation

$\equiv (A \wedge B) \wedge (\neg(\neg A) \wedge \neg B)$ De Morgan

$$\equiv (A \land B) \land (A \land \neg B) \quad \text{Double Negation}$$

$$\equiv A \land (B \land A) \land \neg B \quad \text{Associative}$$

$$\equiv A \land (A \land B) \land \neg B \quad \text{Commutative}$$

$$\equiv (A \land A) \land (B \land \neg B) \quad \text{Associative}$$

$$\equiv (A \land A) \land \text{FALSE} \quad \text{Negation}$$

$$\equiv \text{FALSE} \quad \text{Domination}$$

What does it mean that

$\neg((A \land B) \longrightarrow (A \rightarrow B)) \equiv \text{FALSE}$?

What does it mean that

$\neg((A \land B) \longrightarrow (A \rightarrow B))$ is a contradiction?

We can write ~~proposit~~ any proposition only in terms of $\neg$, $\land$ and $\lor$.

ex: $p \oplus q$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

$$p \oplus q \equiv (p \land \neg q) \lor (\neg p \land q)$$

Then prove that it is correct (truth table)