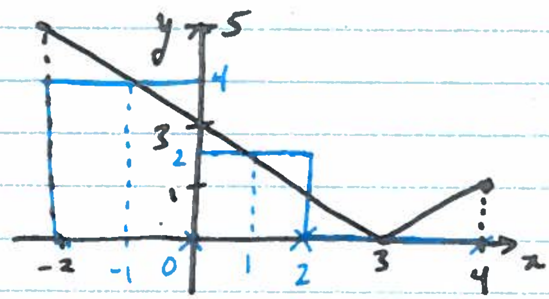
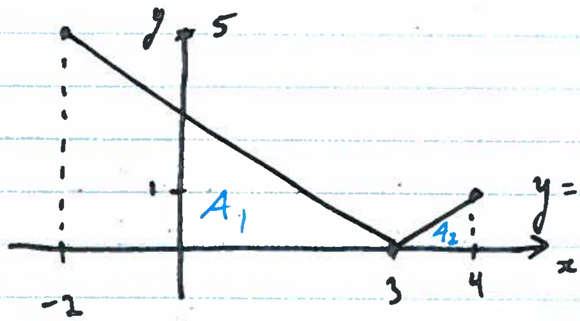
Math 205 - April 2014

① a)



$\Delta x = \frac{4-(-2)}{3} = \frac{6}{3} = 2$ so the partition of $[-2, 4]$ is $\{-2, 0, 2, 4\}$. The midpoints of the three rectangles are, then, $\{-1, 1, 3\}$. So,

$$\begin{aligned} M_3 &= f(-1)\Delta x + f(1)\Delta x + f(3)\Delta x \\ &= |-1-3|\cdot 2 + |1-3|\cdot 2 + |3-3|\cdot 2 \\ &= (4)(2) + (2)(2) + (0)(2) \\ &= 8 + 4 + 0 \\ &= 12. \end{aligned}$$



$$\int_{-2}^{4} f(x)\,dx = A_1 + A_2 = \frac{1}{2}(5)(5) + \frac{1}{2}(1)(1)$$

$$= \frac{25}{2} + \frac{1}{2} = 13$$

Comparison: $\left| \int_{-2}^{4} f(x)\,dx - M_3 \right| = |13 - 12| = 1.$

b) $F'(x) = \frac{d}{dx}\int_{-x}^{x} e^{-t^3}dt = \frac{d}{dx}\left[\int_0^x e^{-t^3}dt - \int_0^{-x} e^{-t^3}dt\right]$

$= e^{-x^3} - e^{-(-x)^3}(-1)$

$= e^{-x^3} + e^{x^3}$

$F'(-2) = e^{-(-2)^3} + e^{(-2)^3} = e^{8} + e^{-8} > 0$

$\therefore$ F is increasing at $x = -2$.

② a) $\int \frac{x^2-3}{x^2-9}dx = \int \frac{x^2-9+9-3}{x^2-9}dx = \int\left(1 + \frac{6}{x^2-9}\right)dx$

$= x + 6\int\frac{dx}{x^2-9}$

$\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3} \Rightarrow 1 = A(x+3)+B(x-3)$

$x = 3 \Rightarrow 1 = 6A \Rightarrow A = 1/6$

$x = -3 \Rightarrow 1 = -6B \Rightarrow B = -1/6$.

$\therefore \frac{1}{x^2-9} = \frac{1}{6(x-3)} - \frac{1}{6(x+3)}$

Then $\int\frac{x^2-3}{x^2-9}dx = x + 6\int\left(\frac{1}{6(x-3)} - \frac{1}{6(x+3)}\right)dx$

$= x + \ln|x-3| - \ln|x+3| + C.$

b) $\int \sqrt{x}\ln x\,dx = \frac{2x^{3/2}\ln x}{3} - \frac{2}{3}\int x^{1/2}dx$

$u = \ln x,\ dv = \sqrt{x}\,dx$

$du = \frac{dx}{x},\ v = \frac{2}{3}x^{3/2}$

$= \frac{2x^{3/2}\ln x}{3} - \frac{4x^{3/2}}{9} + C.$

③ a) $\displaystyle\int \left(t - \frac{1}{t}\right)^2 dt = \int (t^2 - 2 + t^{-2})\, dt$

$$= \frac{1}{3}t^2 - 2t - \frac{1}{t} + C.$$

$F(1) = -1 \Rightarrow \frac{1}{3}(1)^2 - 2(1) - \frac{1}{1} + C = -1$

$$\Rightarrow \frac{1}{3} - 3 + C = -1 \Rightarrow C = \frac{5}{3}$$

$\therefore F(t) = \frac{t^2}{3} - 2t - \frac{1}{t} + \frac{5}{3}.$

b) $\displaystyle\int \frac{e^t}{1+e^{2t}}\, dt = \int \frac{du}{1+u^2} = \arctan u + C = \arctan(e^t) + C.$

$u = e^t \Rightarrow du = e^t dt$

$F(0) = \frac{\pi}{4} \Rightarrow \arctan(e^0) + C = \frac{\pi}{4}$

$$\Rightarrow \arctan(1) + C = \frac{\pi}{4} \Rightarrow \frac{\pi}{4} + C = \frac{\pi}{4} \Rightarrow C = 0.$$

$\therefore F(t) = \arctan(e^t).$

④ a) $\displaystyle\int_0^\pi \cos^2 x \sin^3 x\, dx = \int_0^\pi \cos^2 x \sin^2 x \sin x\, dx$

$\displaystyle = \int_0^\pi \cos^2 x (1 - \cos^2 x) \sin x\, dx = -\int_1^{-1} u^2(1-u^2)\, du$

$u = \cos x \Rightarrow du = -\sin x\, dx$

$x = 0 \Rightarrow u = \cos(0) = 1$

$x = \pi \Rightarrow u = \cos\pi = -1$

$\displaystyle = \int_{-1}^{1} (u^2 - u^4)\, du = \left.\frac{u^3}{3} - \frac{u^{\cancel{3}5}}{5}\right]_{-1}^{1} = \left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{-1}{3} - \frac{-1}{5}\right)$

$= \frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{5} = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$

b) $\displaystyle\int_0^{\pi/4} \sqrt{1+8\tan(x)}\,\sec^2 x\,dx = \int_1^9 u^{1/2}\,\frac{du}{8}$

$u = 1 + 8\tan x \Rightarrow du = 8\sec^2 x\,dx$

$x = 0 \Rightarrow u = 1 + 8(0) = 1$

$x = \pi/4 \Rightarrow u = 1 + 8(1) = 9$

$$= \frac{u^{3/2}}{12}\Bigg|_1^9 = \frac{9^{3/2}}{12} - \frac{1^{3/2}}{12} = \frac{3^3}{12} - \frac{1^3}{12} = \frac{26}{12} = \frac{13}{6}.$$

⑤ $\displaystyle\int_0^4 \frac{1}{x-2}\,dx = \int_0^2 \frac{1}{x-2}\,dx + \int_2^4 \frac{1}{x-2}\,dx.$

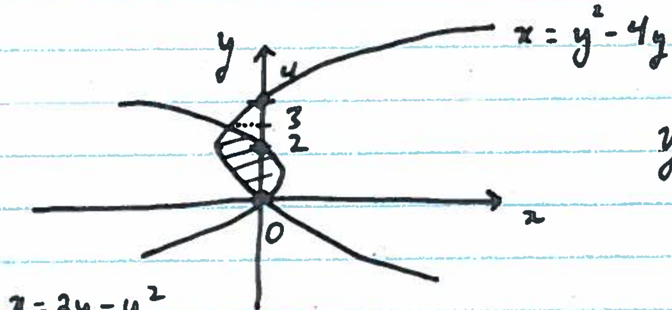
$\displaystyle\int \frac{1}{x-2}\,dx = \ln|x-\overset{2}{\cancel{3}}| + C$

$\displaystyle\int_0^2 \frac{1}{x-2}\,dx = \lim_{t\to 2^-}\int_0^t \frac{1}{x-2}\,dx = \lim_{t\to 2^-} \ln|x-\overset{2}{\cancel{3}}|\Big]_0^t$

$\displaystyle = \lim_{t\to 2^-}\left(\ln|t-\underset{2}{\cancel{3}}| - \ln(2)\right) = -\infty - \ln 2 = -\infty$

There is no need to consider the other integral; we may conclude that $\int_0^4 \frac{1}{x-2}\,dx$ diverges.

⑥ a)



$y^2 - 4y = 2y - y^2$

$0 = 2y^2 - 6y$

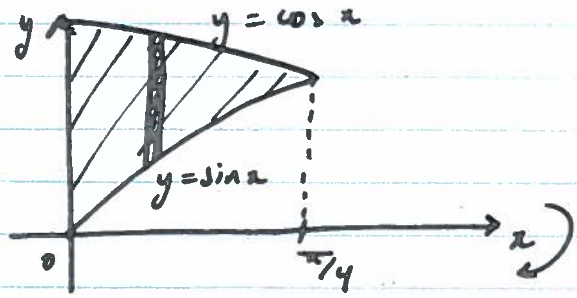
$0 = 2y(y-3)$

$0, 3 = y$

$$A = \int_0^3 \left[(2y - y^2) - (y^2 - 4y)\right] dy = \int_0^3 (-2y^2 + 6y)\,dy$$

$$= \frac{-2}{3}y^3 + 3y^2\Big]_0^3 = -\frac{2(3)^3}{3} + 3(3)^2 = 9$$

b)



$$V = \int_0^{\pi/4} \pi \left[\cos^2 x - \sin^2 x\right] dx$$

method 1: if you recall that $\cos^2 x - \sin^2 x = \cos(2x)$,

$$V = \pi \int_0^{\pi/4} \cos(2x)\, dx = \frac{\pi}{2} \sin(2x) \Big|_0^{\pi/4} = \frac{\pi}{2}\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= \frac{\pi}{2}.$$

method 2: if you don't recall the previous identity,

$$V = \pi \int_0^{\pi/4} \left[\frac{1+\cos(2x)}{2} - \frac{1-\overset{\cos}{\cancel{\sin}}(2x)}{2}\right] dx$$

$$= \pi \int_0^{\pi/4} \left[\frac{1}{2} + \frac{\cos(2x)}{2} - \frac{1}{2} + \frac{\cos(2x)}{2}\right] dx$$

$$= \pi \int_0^{\pi/4} \cos(2x)\, dx = \frac{\pi}{2} \sin(2x) \Big|_0^{\pi/4} = \frac{\pi}{2}\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= \frac{\pi}{2}.$$

c) $$f_{av} = \frac{1}{4}\int_0^4 \frac{x}{\sqrt{9+x^2}}\, dx = \frac{1}{4}\int_9^{25} \frac{u^{-1/2}}{2}\, du = \frac{1}{8} \cdot 2\, u^{1/2} \Big|_9^{25}$$

$u = 9 + x^2 \Rightarrow \frac{1}{2} du = x\, dx$

$x = 0 \Rightarrow u = 9 + 0 = 9$

$x = 4 \Rightarrow u = 9 + 4^2 = 25$

$$= \frac{1}{4}(5-3) = \frac{1}{2}.$$

(7) a) Recall that $\cos(\pi n) = (-1)^n$.

Taking n even, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2}{\sqrt{1+4n^4}} = \lim_{n\to\infty} \frac{3n^2}{\sqrt{n^4\left(\frac{1}{n^4}+4\right)}}$

$= \lim_{n\to\infty} \frac{3}{\sqrt{\frac{1}{n^4}+4}} = \frac{3}{2}$;

taking n odd, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-3n^2}{\sqrt{1+4n^4}} = \lim_{n\to\infty} \frac{-3n^2}{\sqrt{n^4(1/n^4+4)}}$

$= \lim_{n\to\infty} \frac{-3}{\sqrt{\frac{1}{n^4}+4}} = \frac{-3}{2}$.

Therefore, $\lim_{n\to\infty} a_n$ d.n.e.

b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} [\ln(n+1) - \ln(n)] = \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right)$

$= \ln\left(\lim_{n\to\infty} \frac{n+1}{n}\right) = \ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right)$

$= \ln(1+0) = 0$.

(8) a) This is an alternating series with $b_n = \frac{n}{1+n^2}$.

- $\lim_{n\to\infty} \frac{n}{1+n^2} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}+n} = 0$.
- let $f(x) = \frac{x}{1+x^2}$, then $f'(x) = \frac{(1+x^2)-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$

  and $f'(x) < 0 \iff 1-x^2 < 0 \iff 1 < x^2 \iff 1 < |x|$

  $\iff x < -1$ or $x > 1$.

  thus, $\{b_n\}$ is decreasing.

Therefore, by the AST, the series converges.

Consider, now, its series of absolute values $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$.

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges as it's a p-series with $p=1$:

$$\lim_{n\to\infty} \frac{n}{1+n^2} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2}{1+n^2} = \lim_{n\to\infty} \frac{1}{\frac{1}{n^2}+1} = 1.$$

Since $1 \in (0,\infty)$, $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ has the same behavior as $\sum_{n=1}^{\infty} 1/n$; that is, it diverges.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n n}{1+n^2}$ is conditionally convergent.

b) $$L = \lim_{n\to\infty} \left| \frac{(-3)^{3(n+1)}}{(n+1)!} \cdot \frac{n!}{(-3)^{3n}} \right| = \lim_{n\to\infty} \left| \frac{(-3)^3}{n+1} \right|$$
$$= 27 \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

By the Ratio Test, this series converges absolutely (hence, converges).

c) Consider $f(x) = \frac{1}{x(\ln x)^2}$ on $[2,\infty)$.

- $f(x) > 0$ on $[2,\infty)$ since $x>0$ on $[2,\infty)$ and $(\ln x)^2 > 0$ on $[2,\infty)$.
- $f(x)$ is continuous on $[2,\infty)$ since $f(x)$ is only discontinuous at $x = 0, 1$.
- $f(x) = [x(\ln x)^2]^{-1}$
  $$\Rightarrow f'(x) = \frac{-1}{[x(\ln x)^2]^2} \left((\ln x)^2 + 2\ln x \right)$$
  so $f'(x) < 0 \iff (\ln x)^2 + 2\ln x > 0 \iff \ln x(\ln x + 2) > 0$
  since $\ln x > 0$ on $[2,\infty)$, this happens when $\ln x + 2 > 0$,
  i.e. $\ln x > -2 \Rightarrow x > e^{-2}$.

Therefore, $f(x)$ is decreasing on $[2,\infty)$.

Thus, we may apply the Integral Test.

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^2}\,dx \qquad u=\ln x \Rightarrow du=\frac{dx}{x}$$

$$= \lim_{t\to\infty}\int_{\ln 2}^{\ln t} u^{-2}\,du$$

$$= \lim_{t\to\infty} \left.\frac{-1}{u}\right|_{\ln 2}^{\ln t} = \lim_{t\to\infty}\left(\frac{1}{\ln 2}-\frac{1}{\ln t}\right)$$

$$= \frac{1}{\ln 2}.$$

Since the improper integral $\int_2^\infty \frac{1}{x(\ln x)^2}dx$ converges, we have that $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ converges. Since this is a series of positive terms, the convergence is automatically absolute.

9 $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(4x-2)^{n+1}}{n+2}\cdot\frac{n+1}{(4x-2)^n}\right|$

$$= |4x-2|\lim_{n\to\infty}\frac{n+1}{n+2} = |4x-2|$$

$L<1 \Rightarrow |4x-2|<1 \Rightarrow -1<4x-2<1$

$\Rightarrow 1<4x<3 \Rightarrow \frac{1}{4}<x<\frac{3}{4}.$

By the Ratio Test, the series converges for $x\in(1/4, 3/4)$.

When $x = 1/4$, we have $\sum_{n=1}^\infty \frac{(-1)^n}{n+1}$.

This converges by the AST since $b_n = \frac{1}{n+1}$ satisfies

- $\lim_{n\to\infty}\frac{1}{n+1} = 0,$

- $n+1>n \Rightarrow n+2>n+1 \Rightarrow \frac{1}{n+2}<\frac{1}{n+1} \Rightarrow b_{n+1}<b_n.$

When $x = 3/4$, we have $\sum_{n=1}^{\infty} \frac{1}{n+1}$.

Comparing with $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges since it's a power series with $p=1$,

$$\lim_{n\to\infty} \frac{1}{n+1} \cdot n = \lim_{n\to\infty} \frac{n}{n+1} = 1 \in (0, \infty).$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges.

Therefore, the interval of convergence is $[1/4, 3/4)$.

(10) a) $$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{8^{n+1}} \cdot \frac{8^n}{x^{2n+1}} \right|$$
$$= x^2 \lim_{n\to\infty} \frac{1}{8} = \frac{x^2}{8}$$

$$L<1 \Rightarrow \frac{x^2}{8} < 1 \Rightarrow x^2 < 8 \Rightarrow |x| < \sqrt{8} = 2\sqrt{2}.$$

Thus $R = 2\sqrt{2}$.

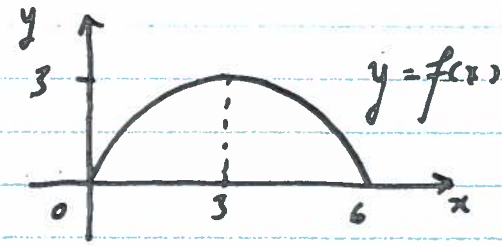
b) $$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{8^n} = x \sum_{n=0}^{\infty} \left(\frac{x^2}{8}\right)^n = x \frac{1}{1-\frac{x^2}{8}} = \frac{8x}{8-x^2}.$$

valid for $|x^2/8| < 1$, i.e. $|x| < 2\sqrt{2}$

Bonus:

a) $f(x) = \sqrt{6x - x^2} = \sqrt{9-(x-3)^2}$ by completing the square.

This is a circle of radius 3 centered at $(3,0)$:



Its domain is [0, 6].

b) $\int_0^6 f(x)\,dx = \frac{1}{2}\pi(3)^2 = \frac{9\pi}{2}$.