DEFFINITIONS AND TERMINOLOGY

DEFFINITION: An Ordinary Differential Equation (**ODE**) is an equation which involves one unknown function (dependent variable), say **y**, and one or more derivatives of y taken with respect to an independent variable, say **x**.

Examples:

$$\frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = 2x$$

$$2x\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + \frac{1}{y} = 2x$$

$$\frac{d^4 y}{dx^4} + x\left(\frac{dy}{dx}\right)^2 - y^2 = \tan x$$

A Partial Differential Equation (**PDE**) is one, which contains partial derivatives.

Examples:

$$\frac{\partial^2 f(x,y,z)}{\partial x^2} + \frac{\partial^2 f(x,y,z)}{\partial y^2} + \frac{\partial^2 f(x,y,z)}{\partial z^2} = 0$$

$$\frac{\partial^2 f(t,r,\theta,z)}{\partial t^2} = const\left(\frac{\partial^2 f(t,r,\theta,z)}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f(t,r,\theta,z)}{\partial \theta^2} + \frac{\partial^2 f(t,r,\theta,z)}{\partial z^2}\right)$$

ORDINARY DIFFERENTIAL EQUATIONS

Consider the following ODE

Derivatives in Leibniz's Notation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Derivatives in Prime Notation

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \ldots + a_1(x)y' + a_0(x)y = g(x)$$

$$where \quad y^n \quad means \quad y^{''''\ldots\ldots} \ {}^{n\ times}$$

If the primes are replaced by dots then we have
The **Derivatives in NEWTON'S dot Notation**

$$a_n(x)\overset{n}{\dot{y}} + a_{n-1}(x)\overset{n-1}{\dot{y}} + \ldots + a_1(x)\overset{o}{y} + a_0(x)\,y = g(x)$$

$$where\ \overset{n}{\dot{y}}\quad means\quad \overset{oo\ldots n\ times}{y}$$

Examples

$$\frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = 2x$$

$$y'' + x\,y' + y = 2x$$

$\ddot{y} + x\dot{y} + y = 2x$ **(traditionally reserved for time derivatives)**

GENERAL FORM OF ODEs

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)\,y - g(x) = 0$$

If *F* is a real valued function given by

$$F(x, y, y', \ldots. y^n) = a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)\,y - g(x)$$

then

$$F(x, y, y', \ldots. y^n) = 0 \qquad\qquad F\quad \text{depends on } n+2 \text{ variables}$$

NORMAL FORM OF ODEs

$$\frac{d^n y}{dx^n} = G(x, y, y', \ldots. y^{n-1})$$

where

$$G(x, y, y', \ldots. y^{n-1}) = -\left\{ \frac{a_{n-1}(x)}{a_n(x)}\frac{d^{n-1} y}{dx^{n-1}} + \ldots + \frac{a_1(x)}{a_n(x)}\frac{dy}{dx} + \frac{a_0(x)}{a_n(x)}y - \frac{g(x)}{a_n(x)} \right\}$$

G depends on *n + 1* variables

Consider the ODE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)\,y - g(x) = 0$$

The order of the highest derivative appearing in the equation it indicates the order of the ODE.

Therefore, the order of the equation under consideration is ***n*.**

The above equation is linear because the power of the dependent variable **y** and **all its derivatives** are **ONE.**

Examples

$$\frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = 2x$$

Linear ODE of order 2

$$2x\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 2x$$

is an ODE of order 2 *and because of* $\left(\frac{dy}{dx}\right)^2$ *it is non linear*

$$2x\frac{dy}{dx} + \frac{1}{y} = 2x$$

is an ODE of order 1 *and because of* y^{-1} *it is non linear*

$$2x\frac{d^4 y}{dx^4} + \sin y = 2x$$

is an ODE of order 4,

due to sin y term the equation is non linear.

Remember that

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \frac{y^9}{9!} - \frac{y^{11}}{11!} + \ldots\ldots\ldots\ldots$$

Therefore, the terms

$$\frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \frac{y^9}{9!} - \frac{y^{11}}{11!} + \ldots\ldots\ldots\ldots$$

make it non linear.
Let us now consider the NORMAL FORM of the equation

$$\frac{d^n y}{dx^n} = G(x, y, y', \ldots y^{n-1})$$

Integrating once we obtain:

$$\frac{d^{n-1} y}{dx^{n-1}} = \int G(x, y, y', \ldots y^{n-1})\, dx + c_1$$

A second integration will result into:

$$\frac{d^{n-2} y}{dx^{n-2}} = \iint G(x, y, y', \ldots y^{n-1})\, dx\, dx + c_1 x + c_2$$

while a third integration will give:

$$\frac{d^{n-3} y}{dx^{n-3}} = \iiint G(x, y, y', \ldots y^{n-1})\, dx\, dx\, dx + \frac{c_1}{2} x^2 + c_2 x + c_3$$

In order to obtain a pure y on the LHS one must perform n integrations. This will produce n constants of integration.

The solution of any differential equation will yield a number of constants that is equal to the order of the equation.

SOME SPECIAL TYPES OF DIFFERENTIAL EQUATIONS

Consider the simple differential equation

$$\frac{dy}{dx} = G(x)$$

Its solution is

$$\int \frac{dy}{dx} dx = \int G(x)\, dx + c_1 \qquad\qquad y = \int G(x)\, dx + c_1$$

The equation

$$\frac{d^2 y}{dx^2} = G(x)$$

can also been solved easily

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = G(x)$$

Integrate once

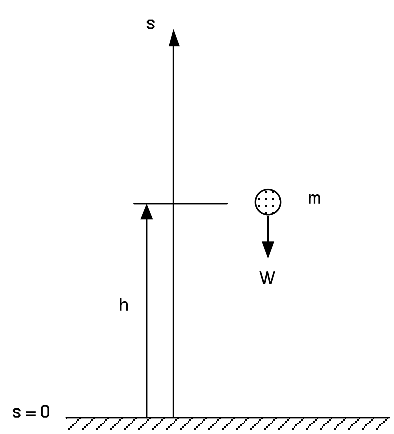
$$\int \frac{d}{dx}\left(\frac{dy}{dx}\right) dx = \int G(x)\, dx + c_1 \qquad \frac{dy}{dx} = \int G(x)\, dx + c_1$$

A second integration yields

$$\int \left(\frac{dy}{dx}\right) dx = \iint G(x)\, dx\, dx + c_1 x + c_2 \qquad y = \iint G(x)\, dx\, dx + c_1 x + c_2$$

Elaboration on Newton's 2nd law using ODEs

Example (Newton's 2nd law) Consider the solid body of mass m in vacuum that $t < 0$ is at rest h units from the ground, as shown in the schematic given below. At $t = 0$ the body is release. Describe the physics following its released from its original position.



Acccording to Newton's 2nd Law:

The change in momentum of the body (m V) = the summation of all the external forces acting on the body (F)

$$\frac{d}{dt}(mV) = F \qquad m\frac{dV}{dt} = W = -mg \quad or \quad \frac{dV}{dt} = -g$$

or

$$\frac{dV}{dt} = -g$$

Therefore,

$$V = -g \int dt + c_1 \qquad V = -g\,t + c_1$$

At $t = 0$ the body is at rest and therefore $V = 0$ and hence

$$0 = -g\,(0) + c_1 \qquad c_1 = 0 \ \text{ or } \ V = -g\,t$$

Upon release the solid will increase its velocity in a linear fashion. Because the last equation does not involve the mass m, any body irrespective of its size and total weight will develop the same velocity upon release.

Since

$$V = \frac{ds}{dt} = -g\,t$$

then

$$s = -g \int t\,dt + c_2 \qquad s = -\frac{g}{2} t^2 + c_2$$

At $t = 0$ the body is located at $s = h$. Therefore,

$$h = -\frac{g}{2} 0^2 + c_2 \qquad c_2 = h \ \text{ or } \ s = -\frac{g}{2} t^2 + h$$

The time required (t_f) for the solid to hit the ground ($s = 0$) is

$$0 = -\frac{g}{2} t_f^2 + h \qquad t_f = \sqrt{\frac{2\,h}{g}}$$

Once more, since the mass is not involved, a large boulder and a feather (***IN VACUUME***), release from the same height at the same time will strike the ground simultaneously.

Another example of an equation, which can be solved easily, is

$$\frac{dy}{dx} = a\,y + b \quad a \quad 0$$

$$Let\ u = ln(a\,y + b) \qquad \frac{du}{dx} = \frac{1}{(a\,y + b)} a \frac{dy}{dx} \qquad \frac{dy}{dx} = \frac{a\,y + b}{\mathrm{a}} \frac{du}{dx}$$

$$\frac{ay+b}{a}\frac{du}{dx} = ay+b \qquad \frac{du}{dx} = a \qquad u = ax + c_1$$

$$ln(ay+b) = ax + c_1 \qquad e^{ln(ay+b)} = e^{ax+c_1} = e^{ax}e^{c_1}$$

$$ay+b = Ce^{ax} \qquad y = \frac{1}{a}\{Ce^{ax} - b\} \qquad or \qquad y = De^{ax} - \frac{b}{a}$$

EQUATIONS WITH VARIABLES SEPARABLE

If a first order differential equation is of the form:

$$\frac{dy}{dx} = f(x)\,g(y)$$

Then one can say that the **variables in the equation are separable**. That is one can write

$$\frac{dy}{g(y)} = f(x)\,dx$$

Then

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + c_1$$

The variables in the previous equation **are separable.**

$$f(x) = 1 \quad and \quad g(y) = ay + b$$

$$\int \frac{dy}{ay+b} = \int (1)\,dx + c_1$$

$$\int \frac{dy}{ay+b} = \frac{1}{a}\,ln\,(ay+b) \qquad and \qquad \int (1)\,dx = x$$

$$\frac{1}{a}\,ln\,(ay+b) = x + c_1 \qquad ln\,(ay+b) = ax + ac_1$$

$$e^{ln(ay+b)} = e^{ax+c_1} = e^{ax}e^{c_1}$$

$$ay+b = Ce^{ax} \qquad y = \frac{1}{a}\{Ce^{ax} - b\} \qquad or \qquad y = De^{ax} - \frac{b}{a}$$