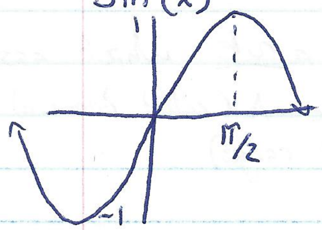
Calculus Midterm #1

Functions

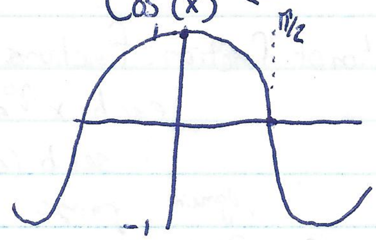
Sin(x) shifted by $\frac{\pi}{2}$

Sin (x)



[Odd]

Cos (x)



[even]

$\cot(x) = \frac{\cos(x)}{\sin(x)}$

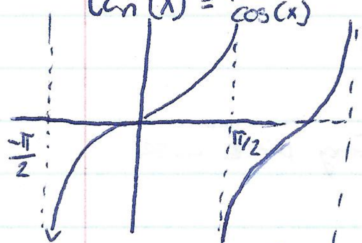
$\sec(x) = \frac{1}{\cos(x)}$

$\csc(x) = \frac{1}{\sin(x)}$

Sin(x) is periodic (ie. a repeating function) of period 2π

A function is even if $f(x) = f(-x)$ (ex: $\cos(x)$) ∴ it is symmetric over y-axis
A function is odd if $f(-x) = -1 \cdot f(x)$ (ex: $\sin(x)$) ∴ it is symmetric about the origin.
A function is neither if it does not fit the definition of even nor odd.

$\tan(x) = \frac{\sin(x)}{\cos(x)}$



· period of π

Domain of $\tan(x) = \{x \in \mathbb{R} \mid x \neq \frac{\pi}{2} + k\pi$, for any $k \in \mathbb{Z}\}$ (| = such that; $\mathbb{Z}$ = integer)

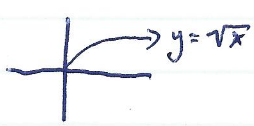
range of $\tan(x) = \mathbb{R}$

image of sin is $[-1, 1]$ $(= \{x \in \mathbb{R} \mid -1 \leq x \leq 1\})$
↳ y-values that are actually obtained

range = what kinds of objects are the output of the function. For us, range are rational #

For image Notation
[,] = including those values, like ←●——●→
(,) = excluding those values, like ←○——○→
* Do not include ∞, it is not a number

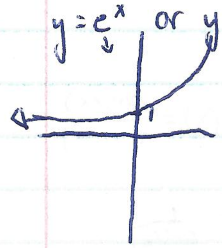
ex
$f(x) = \sqrt{x}$
domain $[0, \infty)$
image $[0, \infty)$



For radical functions (ex: $\frac{x+2}{x^2-1}$)
domain is every where where $x^2 - 1 \neq \emptyset$

Exponential functions

$y = e^x$ or $y = a^x$



domain $\mathbb{R}$

image $(0, \infty)$

log functions

$y = \ln(x) \iff x = e^y$

$y = \log_a x \iff x = a^y$

Definition of function: function has is a <u>rule</u> which assigns to each x in a set A (called domain) a set b (called range)

Notation: $\underset{\text{name}}{f} : \overset{\text{domain}}{\mathbb{R}} \to \overset{\text{range}}{\mathbb{R}}$

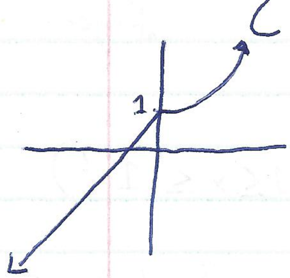
$\underbrace{x \to x^2}_{\text{how to calculate}}$

<u>Combining functions</u>

1) Piecewise Functions

"splice functions together"

ex: $f(x) = \begin{cases} x+1 & \text{if } x \leq 0 \\ e^x & \text{if } x > 0 \end{cases}$



2) Absolute Value Functions*

$y = |x|$

$\to y = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

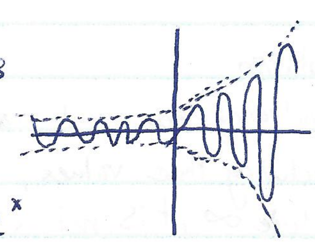
* Fancy piece wise

3) Sum, difference, product, Quotient of functions

$y = f(x) + 2 \leftarrow$ raise by 2u.

$y = 2 \cdot f(x) \leftarrow$ scale by 2

$h(x) = f(x) \cdot g(x)$, $f(x) = \cos(x)$, $g(x) = e^x$



4) Composition of Functions

$(f \circ g)(x) = f(g(x))$

$\neq f(x) \cdot g(x)$

Calculus mid term #1

5) Inverse function

definition: A function is invertable if, for every y in the image of f, there is a unique x in the domain of f

- passes the horizontal line test (1 x to 1 y for every y)
- must already pass the vertical line test (to be function)

* if it does not pass the HLT, it might if you restrict the o.g. graph.

Try!

$f'(x)$ of $\frac{x-2}{x+3}$?

$$y = \frac{x-2}{x+3}$$
$$(x+3)y = x-2$$
$$3y + xy = x-2$$
$$xy - x = -2-3y$$
$$x(y-1) = -2-3y$$
$$x = \frac{-2-3y}{y-1}$$
$$f'(x) = \frac{-2-3x}{x-1}$$
$$x \neq 1$$

Key properties

$y = e^x \Leftrightarrow x = \ln y$

$e^{\ln(x)} = x \quad (x > \emptyset)$

$\ln(e^x) = x \quad (x \in \mathbb{R}$

$y = 2^x = e^{x\ln(2)} \longrightarrow y = 2^x$

$$\ln(y) = \ln(2^x)$$
$$\ln(y) = x\ln(2)$$
$$e^{\ln(y)} = e^{x\ln(2)}$$
$$y = e^{x\ln(2)}$$

$$\log_2 x = \frac{\ln(x)}{\ln(2)}$$

Log rules

$$\ln(x_1 \cdot x_2) = \ln(x_1) + \ln(x_2)$$
$$\ln(x_1^{x_2}) = x_2\ln(x_1)$$
$$\ln\left(\frac{x_1}{x_2}\right) = \ln(x_1) - \ln(x_2)$$

Exponential Rules

$$a^{x_1+x_2} = a^{x_1} \cdot a^{x_2}$$
$$a^{x_1-x_2} = a^{x_1} \div a^{x_2}$$
$$(a^{x_1})^{x_2} = a^{x_1 \cdot x_2}$$
$$(a \cdot b)^x = a^x \cdot b^x$$

To inverse $\sin(x)$, you need to restrict the domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $\sin(x)$ inverted is $\arcsin(x)$

The inverse of $y = e^x$ is $y = \ln(x)$

$y = a^x$ is $y = \log_a x$

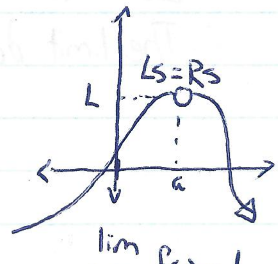
Idea of slope of tangent line

$$\lim_{h \to \emptyset} \frac{f(a+h) - f(a)}{h} = \frac{\Delta y}{\Delta x}$$

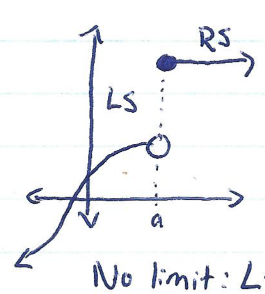
when solving do not just plug in numbers

"Pimple popping"

Limit definition: Suppose f is a function on a domain around the point $x=a$ (can exclude a). Then we write $\lim_{x \to a} f(x) = L$. If



$\lim_{x \to a} f(x) = L$



No limit: $LS \neq RS$

Limit Laws

Suppose $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$

Add: - $\lim_{x \to a} (f(x) + g(x)) = L + M$
Sub: $-$ $-$

mult: - $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M$
div $\div$ $\div$

with a constant: $\lim_{x \to a} c \cdot f(x) = C \cdot L$

One-sided Limits

Left-hand limit: make f(x) as close to L as possible by restricting to $x < a$ ~~a~~

$\lim_{x \to a^-} f(x) = L$

left side

Right-hand limit: f(x) gets close to L as $x \to a$ from right

$\lim_{x \to a^+} f(x) = L$

Example

$$f(x) = \begin{cases} x+2 & \text{if } x < 5 \\ 17 & \text{if } x = 5 \\ x^2 - 3x & \text{if } x > 5 \end{cases} \qquad \lim_{x \to 5} ?$$

LS

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (x+2)$

$= 7$

RS

$\lim_{x \to 5^+} f(x) = x^2 - 3x$

$= 25 - 15$

$= 10$

$7 \neq 10$

$LS \neq RS$

$\therefore$ The limit does not exist

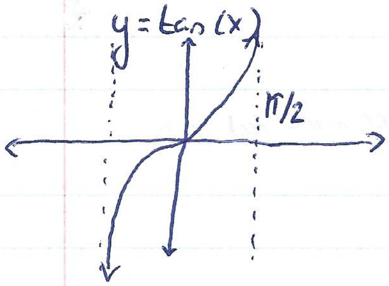
Continuity

If a is in the domain of f and $\lim_{x \to a} f(x) = f(a)$ then f is continuous at a

The following functions are continuous e^x, $\ln(x)$, x^n, $x^{1/n}$, x^t, $\sin(x)$, $\cos(x)$ $t \in \mathbb{R}$
(root functions)

- polynomial + rational functions.

Infinite limits = those that have a value of $\pm\infty$, where the function grows without bound as it approaches some value a.
*Caused by vertical asymptote.



$$\left[\lim_{x\to 0} \frac{x}{x} = 1 \;\middle|\; \lim_{x\to 0} \frac{x^2}{x} = \emptyset \;\middle|\; \lim_{x\to 0^+} \frac{x}{x^2} = \emptyset \right]$$

To summarize; A function is continuous at a point if...
1) a (the point) is in the function
2) $\lim_{x\to a} f(x)$ exists
3) $f(x) = f(a)$

Squeeze Theorem

suppose f, g, h are 3 functions such that
$g(x) \le f(x) \le h(x)$ near $x=a$
so $\lim_{x\to a} f(x) = b$

$$\lim_{x\to a} g(x) = b = \lim_{x\to a} h(x)$$

↳ Example
$f(x) = x^2 \cdot \sin\frac{1}{x}$
what is $\lim_{x\to\emptyset}$?
$g(x) = -x^2$
$h(x) = x^2$

$-x^2 \le x^2 \cdot \sin\frac{1}{x} \le x^2$
$-1 \le \sin\frac{1}{x} \le 1$
$\therefore \lim_{x\to 0} x^2 \sin\frac{1}{x} = \emptyset$

$\lim_{x\to 0} x^2 = \emptyset$
$\lim_{x\to\emptyset} -x^2 = \emptyset$

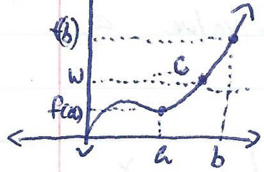
* sosmath.com on squeeze theorem

Method to find Limit
1) if $f(x)$ is continuous on both sides, plug in a.
2) if $f(x)$ has continuous functions on both sides, use one-sided limits + plug in a
3) if ans is $\frac{\text{constant}}{0}$, decide if $\pm\infty$ (using 1-sided)
4) if $0/0$ or ∞/∞ "indeterminate forms" factor, expand, simplify, rationalize ...

Composition with a continuous function
f is continuous at b & $\lim_{x\to a} g(x) = b$
then $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b)$

Floor theorem
$\lfloor x \rfloor$ = closest integer which satisfies
$n \le x$

Intermediate Value theorem



- If we know the function is continuous and that it has $f(a)$ and $f(b)$ then we know it must cross line w at some point $f(c) = w$

* useful for determining roots.

Example: Show that $f(x) = x^6 + 4x^5 - 5x^4 + x + 1$ has 1 or more real roots.

$f(0) = 1$

$f(1) = 1 - 4 - 5 - 1 + 1$
$= -8$

Since $f(x)$ is continuous & below x-axis at $f(-1)$ & above at $f(\emptyset)$, it must have 1 root between them.

$\lim_{x \to \infty} f(x)$ "Horizontal asymptotes"

Method: Divide by highest x^n in denom.

→ example $\lim_{x \to \infty} \dfrac{x^2 - x}{3x^2 + 1}$

$= \lim_{x \to \infty} \dfrac{x(x+1)}{3x^2+1} \div \dfrac{x^2}{x^2}$

$= \lim_{x \to \infty} \dfrac{\left(\frac{x-1}{x}\right)}{3 + \frac{1}{x^2}}$

$= \lim_{x \to \infty} \dfrac{1 - \left(\frac{1}{x}\right) \to \emptyset}{3 + \left(\frac{1}{x^2}\right) \to \emptyset}$ as $x \to \infty$

$= \dfrac{1}{3}$

Derivatives

$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ whenever this limit exists

ex: $f(x) = \frac{1}{x}$

$= \lim_{h \to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$

$= \lim \left[\dfrac{x - (x+h)}{x^2 + xh}\right] \cdot \dfrac{1}{h}$

$= \lim \left[\dfrac{-h}{x^2+xh}\right] \cdot \dfrac{1}{\cancel{h}}$

$= \lim_{h \to \emptyset} \dfrac{-1}{x^2 + xh}$

$f'(x) = \dfrac{-1}{x^2}$

* If a function is differentiable at x then F is continuous at x.

If f is discontinuous at x, you can't $f'(x)$ it.

A function can be continuous but not differentiable

$f'(x^2)$ is $2x$

$f'(\sqrt{x})$ is $\frac{1}{2\sqrt{x}}$

$f'(\frac{1}{x})$ is $\frac{-1}{x^2}$

Calculus

Power Rule

No x's, that's exponential

$f'(x^n) = n \cdot x^{(n-1)}$

Exponential Functions

$f(x) = e^x$
$f'(x) = e^x$

$f(x) = e^{g(x)}$
$f'(x) = e^{g(x)} \cdot g'(x)$

$f(x) = b^x$
$f'(x) = b^x \cdot \ln b$

$f(x) = a^{nx}$
$f'(x) = n \cdot a^{nx} \cdot \ln a$

$f(x) = b^{g(x)}$
$f'(x) = b^{g(x)} \cdot \ln b \cdot g'(x)$

$f(x) = b^{n \cdot g(x)}$
$f'(x) = n \cdot b^{g(x)} \cdot \ln b \cdot g'(x)$

Trig identities

$\csc\theta = \frac{1}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$

$\cot\theta = \frac{1}{\tan\theta}$

$\cos^2(x) + \sin^2(x) = 1$

Linearity of the derivative

Let f & g be differentiable functions and let $c \in \mathbb{R}$ (a constant).

Then $\frac{d}{dx} c\, f(x) = c \cdot \frac{d}{dx} f(x)$

ie. $\frac{d}{dx} 3x^3 = 3 \cdot \frac{d}{dx} x^3 = 3 \cdot (3x^2) = 9x^2$

Product rule

$\frac{d}{dx}(f \cdot g)(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Quotient Rule

$\frac{d}{dx} f(x) \div g(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$

Sine & cosine & tan

$\frac{d}{dx} \sin(x) = \cos(x)$

$\frac{d}{dx} \cos(x) = -\sin(x)$

$\frac{d}{dx} \tan(x) = \sec^2(x)$

$\frac{d}{dx} \csc(x) = -\csc(x) \cdot \cot(x)$

$\frac{d}{dx} \cot(x) = -\csc^2(x)$

→

Chain Rule

$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$