Bayes' Theorem

Lecture Goals

After completing this lecture, you should be able to:

- Use Bayes' Theorem for conditional probabilities

Bayes' Theorem I

- A method of revising conditional probabilities by using available information

- It enables you, to find the probability of B given A when the probability of A given B is known

- Based on the definition of conditional probability and the law of total probability

Bayes' Theorem II

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Applying the law of total probability to the denominator

$$= \frac{P(A \cap B)}{P(A \cap B) + P(A \cap \bar{B})}$$

Applying the definition of conditional probability throughout

$$= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$$

Prior & Posterior Probability

- Note:
  - P(B) is called the prior probability
    - the probability of B before information about the event A is given
  - P(B|A) is called the posterior probability
    - the probability of B after information about the occurrence of A is given

Bayes' Theorem III

- If B_1, B_2, . . ., B_k are mutually exclusive then for any event A in sample space S

$$P(A) = \sum_{i=1}^{k} P(A \mid B_i) \cdot P(B_i)$$

$$= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \ldots + P(A|B_k)P(B_k)$$

Bayes' Theorem IV – General

- If B_1, B_2, . . ., B_k are mutually exclusive then…

$$P(B_1|A) = \frac{P(A \cap B_1)}{P(A)}$$

$$= \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \ldots + P(A|B_k)P(B_k)}$$

Example I: Bayes' Theorem

- When an auditor checks a small company's accounts, he has an initial idea of the probability of an error in the accounts based on other accounts he's audited in the past. Call this $P(\text{error}) = 0.05$. However, auditors can make mistakes. They may audit perfectly good accounts and think there is an error, $P(report\ an\ error|no\ error) = 0.04$. This is called a false positive. Or they may find an error that is in fact there $P(report\ an\ error|error) = 0.94$. Sometimes they may fail to find the error $P(report\ no\ error|error) = 0.06$ - a false negative. Suppose the auditor reports an error and we would like to know the probability, $P(error|report\ an\ error)$, that there is in fact an error in the accounts.

Example II: Bayes' Theorem

- A drilling company has estimated a 40% chance of striking oil for their new well.
- A detailed test has been scheduled for more information. Historically, 60% of successful wells have had detailed tests, and 20% of unsuccessful wells have had detailed tests.
- Given that this well has been scheduled for a detailed test, what is the probability that the well will be successful?

Example II: Bayes' Theorem

(continued)

- Let S = successful well

  U = unsuccessful well

- P(S) = .4 , P(U) = .6 (prior probabilities)
- Define the detailed test event as D
- Conditional probabilities:

$$P(D|S) = .6 \qquad P(D|U) = .2$$

- Goal is to find P(S|D)

Example II: Bayes' Theorem

(continued)

Apply Bayes' Theorem:

$$P(S|D) = \frac{P(D|S)P(S)}{P(D|S)P(S) + P(D|U)P(U)}$$

$$= \frac{(.6)(.4)}{(.6)(.4) + (.2)(.6)}$$

$$= \frac{.24}{.24 + .12} = .667$$

So the revised probability of success (from the original estimate of .4), given that this well has been scheduled for a detailed test, is .667

Example III: Bayes' Theorem

- The Watts New Light bulb Corporation ships large consignments of light bulbs to big industrial users. When the production process is functioning correctly, which is 90% of the time, 10% of all light bulbs produced are defective. However, the process is susceptible to an occasional malfunction, leading to a defective rate of 50%.
  a. If a defective bulb is found, what is the probability that the process is functioning correctly?
  b. If a non-defective bulb is found, what is the probability that the process is operating correctly?

Lecture Summary

- Discussed Bayes' Theorem and applications