SYSTEMS OF LINEAR EQUATION L1

→ A linear equation in n variables $x_1, x_2 \ldots x_n$ is an equation that can be written in the form $a_1x_1 + a_2x_2 \ldots\ldots + a_nx_n = b$, (Where the coefficients $a_1, a_2 \ldots a_n$ and b are constants

$y = mx + b$ $\sqrt{8}x + y = 6$ (linear equation) The power on x has to 1

$y - mx = b$

$a_1 = 1$

$a_2 = -m$

$a_3 = b$

→ Solution of a linear Equation

$a_1x_1 + a_2x_2 \ldots\ldots + a_nx_n = b$ is a vector $[s_1, s_2 \ldots s_n]$ whose components satisfy the equation when we solsitute

$x_1 = s_1$, $x_2 = s_2$, $x_n = s_n$

→ For example

$2x_1 - x_2 - 3x_3 = -1$ is $(1,2,1)$ a solution?

$2 - 2 - 3 = -3 \neq -1$

$(1,0,1)$

$2(1) - 0 - 3(1) = -1 = -1$

→ A system of linear equations (or a linear system) is a finite set of linear equations envolving the same set of variables say $x_1, x_2 \ldots x_n$

→ For example

$2x_1 - x_2 - 3x_3 = -1$

$-2x_1 + 2x_2 + 5x_3 = 3$

is a system of linear equation with 3 variables (x_1, x_2, x_3) and 2 linear eq/ns

A $\left.\begin{matrix} x^2 + y = 7 \\ x - 5y = 10 \end{matrix}\right\} \times$ B $\left.\begin{matrix} 3x + xy = 6 \\ x - y = 3 \end{matrix}\right\} \times$ C $\left.\begin{matrix} \sqrt{8}x + y = 4 \\ 2x - 3y = 5 \end{matrix}\right\} \checkmark$

→ The set of all possible solutions to a linear system is called the solution set of the system

$2x_1 - x_2 - 3x_3 = -1$

$-2x_1 + 2x_2 + 5x_3 = 3$

(1,0,1) & (2,-4,3) are solution? check?

$(2)-(0)-(1) = -1 = -1$ | $(4)+(4)-(9) = -1 = -1$

$-2+(0)+5 = 3 = 3$ | $(-4)-(8)+(15) = 3 = 3$

(1,0,1), (2,-4,3) form a solution set to the linear system

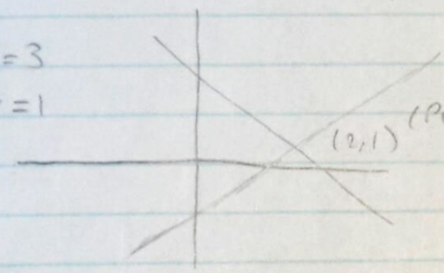
→ Two linear systems are called equivalent if they have the solution set.

$\left.\begin{array}{l} x+y=4 \\ x-y=2 \end{array}\right\}$ $\left.\begin{array}{l} 8x_1+2y_1=26 \\ 13x_1+3y_1=42 \end{array}\right\}$

(3,1) is the solution set for both systems, so the two systems are equivalent geometrically, the solution of two linear equations in two variables is the intersection of two lines.

① L_1 $x+y=3$

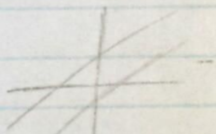
L_2 $x-y=1$



(Point of intersection) (Unique solution)

② L_1 : $-x+2y=1$

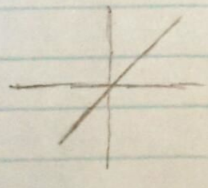
L_2 : $x-2y=-3$



(These two lines are parallel therefore No solution to the linear system)

③ L_1 $2x-y=1$

$-4x+2y=-2$



L_1 and L_2 coincide so they are infinitely many solutions

→ A system of linear equations has either
- No solution (Inconsistent linear system)
- Unique solution (consistent linear system)
- Infinitely many solutions

L2

$$\left.\begin{aligned} x - 2y + z &= 0 \\ 2y - 8z &= 8 \\ -4x + 4y + 9z &= -9 \end{aligned}\right\}$$ 3 variables x y and z, 3 equations

$$\begin{array}{c} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 4 & 9 \end{bmatrix} \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$ is the coefficient matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 4 & 9 & -9 \end{array}\right]$$

→ A matrix with m rows and n columns has size $m \times n$. Coefficient matrix is 3×3

Row Echelon Form: (REF)

→ A leading entry of a row refers to the left most, non-zero entry (in a non-zero row)

→ Definition: A matrix in row echelon form (REF) if it satisfy
1) A row consisting of entirely zeros are at the bottom
2) In each non-zero row; the leading entry of the row is in a column to the left of any leading entry below it

Reduced Row Echelon Form (RREF)

→ A matrix in RREF if it satisfies:
1) It is in REF form
2) The leading entry in each row is 1
3) Each leading 1 is the only non-zero entry.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ REF RREF

Elementary Row operations

→ Any matrix can be reduced to a matrix in REF using elementary row operations
1) Interchanging any two rows
2) Multiply a row by a constant
3) Replace a row by the sum of itself and a multiple of another row.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 5 & 6 \end{bmatrix}$$

a) $R_1 \leftrightarrow R_3$
b) $R_2 = 2R_2$
c) $R_3 = R_3 + 3R_2$

a) $\begin{bmatrix} 4 & 5 & 6 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 4 & 5 & 6 \end{bmatrix}$ c $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 8 & 12 \end{bmatrix}$

L3

→ A pivot position of a leading entry in an echleon form of a matrix
→ A pivot is a non-zero number that is a pivot position
→ A pivot is a column that contains a pivot position
→ A basic variable is a variable that corresponds to a pivot column.
→ A free variable is a variable that is not a basic variable i.e they correspond to a non-pivot column.

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array}$$
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

(1,1) and (2,3) are pivot positions
1 and 4 are pivots.
column 1 and column 3 are pivots.

x_1 and x_3 are basic variable
x_2 is a free variable

→ Solving a linear system.
→ To solve a linear system (system of linear equations) we set up the corresponding augmented matrix, and use elementary row operations to reduce the matrix
→ We work by column from left to right and top to bottom. The strategy is to create a leading entry in a column and then use it to create zeros below it.

$$\begin{aligned} x + 2y &= 4 \\ -x + 3y + 3z &= -2 \\ y + z &= 0 \end{aligned}$$

3 variables
3 equations

→ The corresponding augmented matrix is

REF

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ -1 & 3 & 3 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 5 & 3 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 3 & 2 \end{array}\right] \xrightarrow{R_3' = R_3 - 5R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 2 \end{array}\right]$$

Choice 1 → Stop when you have REF and use back subsitution to ~~find~~ solve the system

$z = -1$
$y = 1$
$x = 2$

solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

Choice 2

Continue row operation until RREF

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right] \xrightarrow{R_3 = R_3 \div 2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_2 = R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1 = R_1 - 2R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow \text{solution is } (x, y, z) = (2, 1, -1)$$

→ REF of a matrix is not unique (Everyone will have different REF)
→ RREF is unique (All will have the same RREF)

E1)

$$\begin{aligned} x + 2y + z &= 3 \\ x - y + z &= 1 \\ -2x - 4y - 2z &= 4 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ -2 & -4 & -2 & 4 \end{array}\right] \xrightarrow[R_3' = R_3 + 2R_1]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & 0 & 10 \end{array}\right]$$

$$\underline{0 = 10}$$

∴ The system of linear equation is inconsistent (No solutions)

E2)

$$\begin{aligned} x + 2y - 3z &= 3 \\ -2x - 5y + 4z &= 5 \\ -5x - 13y + 9z &= 18 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ -2 & 5 & 4 & 5 \\ -5 & -13 & 9 & 18 \end{array}\right] \xrightarrow[R_3' = R_3 + 5R_1]{R_2' = R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 9 & -2 & 11 \\ 0 & 3 & -6 & 22 \end{array}\right]$$

$$\xrightarrow{R_3' = R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2' = -R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & 1 & 2 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 25 \\ 0 & 1 & 2 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

Column 1 and column 2 are pivot columns
x and y are basic variables and z is a free variable
$z = t \quad (t \in \mathbb{R})$
(From eq, 2) $y = -11 - 2t$
(From eq, 1) $x = 25 + 7t$

$$\begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 25 \\ -11 \\ 0 \end{bmatrix}$$

Find the value of the constant K such that the following system has

i) No solution

ii) Infinitely many solutions

iii) Unique solution

$$x + 2y - z = 1$$
$$-2x - 3y + 2z = -1$$
$$-5x - 8y + 5z = k$$

1)

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -2 & -3 & 2 & -1 \\ -5 & -8 & 5 & K \end{array}\right] \begin{array}{c} R_2' = R_2 + 2R_1 \\ R_3' = R_3 + 5R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 5+K \end{array}\right] \; R_3' = R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3+K \end{array}\right]$$

1) No solution when $3+K \neq 0$

$K \neq -3$

2) Infinitely many solutions if $K = -3$

3) No unique solution possible

$x + hy = 1$

$(3-3h)y = K-2$

$$\left[\begin{array}{cc|c} 1 & h & 1 \\ 0 & 3-3h & K-2 \end{array}\right]$$

1) No solution if $3-3h = 0$ and $K-2 \neq 0$

$h = 1$ and $K \neq 2$

2) Infinitely many solutions when $3-3h = 0$ and $K-2 = 0$

$h = 1$ and $K = 2$

3) Unique solution if $3-3h \neq 0$ $(h \neq 1)$

Solve the following linear system

2) $x + 3y - 2z - w = 0$
$-2x - 5y \quad + 4w = 0$
$x + 4y - 6z + w = 0$

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 0 \\ -2 & -5 & 0 & 4 & 0 \\ 1 & 4 & -6 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + 2R_1 \\ R_3' = R_3 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 1 & -4 & 2 & 0 \end{array}\right] \xrightarrow{R_3' = R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since 1 & 2 are pivot columns the corresponding variable x and y are basic variable z & w are free variables

let $z = t$, $t \in R$
$w = s$, $s \in R$

1) $y - 4z + 2w = 0$
$y = 4z - 2w$
$= 4t - 2s$

$x + 3y - 2z - 1w = 0$
$x = -3y + 2z + w = 0$
$x = -3y + 2t + s$
$x = -3(4t - 2s) + 2t + s$
$= -12t + 6s + 2t + s$
$= -10t + 7s$

the solution is (Parametric Vector form)

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -10t + 7s \\ 4t - 2s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix} s$$

Solve the non-homogenous linear solution.

3) $x + 3y - 2z - w = 2$
$-2x - 5y \quad + 4w = 3$
$x + 4y - 6z + w = 9$

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 2 \\ -2 & -5 & 0 & 4 & 3 \\ 1 & 4 & -6 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 2 \\ 0 & 1 & -4 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t$, $t \in R$
$w = s$, $s \in R$
$y - 4z + 2w = 7$
$y = 7 + 4z - 2w$
$= 7 + 4t - 2s$

$x + 3y - 2z - w = 2$
$x = 2 - 3y + 2z + w$
$x = -3y + 2z + w + 2$
$x = -3(7 + 4t - 2s) + 2t + s + 2$
$x = -21 - \cancel{12t} + 6s + \cancel{2t} + s + 2$
$= -10t + 7s - 19$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -10t + 7s - 19 \\ 4t - 2s + 7 \\ t \\ s \end{bmatrix} = \begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} -19 \\ 7 \\ 0 \\ 0 \end{bmatrix}; \ s \,\&\, t \in \mathbb{R}$$

→ Give the solution to the corresponding homogenous solution just remove the constants

→ General solution of a non-homogenous system = general solution of the corresponding system + a particular solution of the non-homogenous system

$$\begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix} s = V_h \qquad V_h + p \text{ where } p = \begin{bmatrix} -19 \\ 7 \\ 0 \\ 0 \end{bmatrix}$$

Vector addition

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} \quad x + y \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

Scalar Multiplication

$$\text{let } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad cx = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$$

If $\vec{v_1}, \vec{v_2}, \vec{v_p}$ are vectors in $\mathbb{R}^n$, and x_1, x_2 are scalars, the vector $\vec{b}$ vector $\vec{b} = x_1\vec{v_1} + x_2\vec{v_2} + \dots x_p\vec{v_p}$ is called a linear

→ Cobination of $\vec{v_1}, \vec{v_2} \dots \vec{v_p}$ using weights $x_1, x_2 \dots x_p$

$\vec{v_1} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ Is $\vec{b}$ $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ a linear combination?

$1\vec{v_1} + 2\vec{v_2} + (-1)\vec{v_3} = \vec{b}$

$x_1 = 1,\ x_2 = 2,\ x_3 = -1$

→ Determine if the vector $b = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$ is a linear combinations of the vectors

$\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}$ $x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3} = \vec{b}$

$x_1 \begin{bmatrix} x_1 \\ 0 \\ x_1 \end{bmatrix} + \begin{bmatrix} -2x_2 \\ 3x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} -6x_3 \\ 7x_3 \\ 5x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$

$\begin{bmatrix} x_1 - 2x_2 - 6x_3 \\ 3x_2 + 7x_3 \\ x_1 - 2x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$ $\left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{array}\right] \xrightarrow{R_3' = R_3 - R_1} \left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{array}\right]$

$11x_3 = -2$

$x_3 = \frac{-2}{11}$

$3x_2 + 7x_3 = -5$

$3x_2 + 7\left(\frac{-2}{11}\right) = -5$

$3x_2 + \frac{-14}{11} = -5$

$x_2 = -\frac{41}{33}$

$x_1 - 2x_2 - 6x_3 = 11$

$x_1 - 2\left(-\frac{41}{33}\right) - 6\left(\frac{-2}{11}\right) = 11$

$x_1 + \frac{82}{33} + \frac{12}{11} = 11$

$x_1 = 11 - \frac{118}{33}$

$x_1 = \frac{245}{33}$

$\frac{245}{33} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \left(-\frac{41}{33}\right) \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} + \left(\frac{-2}{11}\right) \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}$

$$x_1\begin{bmatrix}1\\-2\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\2\end{bmatrix}+x_3\begin{bmatrix}5\\-6\\8\end{bmatrix}=\begin{bmatrix}2\\-1\\6\end{bmatrix} \quad \begin{bmatrix}1&0&5\\-2&1&-6\\0&2&8\end{bmatrix}=\begin{bmatrix}2\\-1\\6\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&0&5&2\\-2&1&-6&-1\\0&2&8&6\end{array}\right]\underset{R_2'=R_2+2R_1}{\sim}\left[\begin{array}{ccc|c}1&0&5&2\\0&1&4&3\\0&2&8&6\end{array}\right]\underset{R_3'=R_3-2R_2}{\sim}\left[\begin{array}{ccc|c}1&0&5&2\\0&1&4&3\\0&0&0&0\end{array}\right]$$

$x_1=2-5x_3$
$x_2=3-4x_3$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2-5x_3\\3-4x_3\\x_3\end{bmatrix}=\begin{bmatrix}2\\3\\0\end{bmatrix}+\begin{bmatrix}-5\\-4\\1\end{bmatrix} \quad x_3, x_3 \in \mathbb{R}$$

3) $\vec{b}=\begin{bmatrix}h\\-3\\-5\end{bmatrix}$, $\vec{v_1}=\begin{bmatrix}1\\0\\-2\end{bmatrix}$, $\vec{v_2}=\begin{bmatrix}-2\\1\\7\end{bmatrix}$

$$\left[\begin{array}{cc|c}1&-2&h\\0&1&-3\\-2&7&-5\end{array}\right]\underset{R_3'=R_3+2R_1}{\sim}\left[\begin{array}{cc|c}1&-2&h\\0&1&-3\\0&3&-5+2h\end{array}\right]\underset{R_3'=R_3-3R_2}{\sim}\left[\begin{array}{cc|c}1&-2&h\\0&1&-3\\0&0&2h+4\end{array}\right]$$

If $h=-2$ then b is a linear combination of $\{\vec{v_1}, \vec{v_2}\}$

$2h+4=0$
$h=-2$

The set of all linear combination of $\vec{v_1}, \vec{v_2} \ldots \vec{v_p}$ is called span of $\vec{v_1}, \vec{v_2} \ldots \vec{v_p}$ and is denoted

$\vec{v_1}=\begin{bmatrix}1\\0\end{bmatrix}$, $\vec{v_2}=\begin{bmatrix}0\\1\end{bmatrix}$ Span $\{\vec{v_1}, \vec{v_2}\}=\mathbb{R}^2$

Span $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ x axis

→ Is

Linear Algebra

Linear Independence

→ A set of vectors $\{v_1, v_2 \dots v_k\}$ in R^n is said to be linearly independent if the vector equation if the vectors equation $c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_k\vec{v_k} = \vec{0}$

has only trivial solution
i.e. $c_1 = c_2 = \quad c_K = 0$

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

→ There is no $c_1 \neq 0$ or $c_2 \neq 0$ to satisfy the above equation

→ The set $\{v_1, v_2, v_K\}$ in R^n is said to be linearly dependent if there exists scalars $c_1, c_2, \; c_K$ not all zero, such that $c_1v_1 + c_2v_2 + c_Kv_K = \vec{0}$

$$3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \end{bmatrix} \right\}$$

→ Define if the vectors $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix}$ are linearly dependent?

→ Equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 5 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 3 & -5 & 5 & 0 \\ -2 & 6 & -6 & 0 \end{array}\right] \xrightarrow{R_3' = R_3 + 2R_1} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$$

From eq 3, $z = 0$
From eq 2, $4y + 0 = 0$
$y = 0$

From eq 1, $1x \cdot 0 = 0$
$x = 0$

⇒ The given vectors are linearly independent

Decide if the vectors $\begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -2 \\ -6 \end{bmatrix}$ are linearly dependent?

$$\left[\begin{array}{ccc|c} 3 & 4 & 3 & 0 \\ -1 & -7 & 7 & 0 \\ 1 & 3 & -2 & 0 \\ 0 & 2 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = 0, x_2 = 0, x_1 = 0.$

∴ The vectors are linearly independent.

→ If the REF has a pivot in each column then the vectors are linearly independent.

→ The following statements are equivalent for an $m \times m$ matrix A

i) For each b in $\mathbb{R}^n$, the equation $Ax = b$ has a solution

ii) Each b in $\mathbb{R}^n$ is a linear combination of the columns of A

iii) The columns of A span $\mathbb{R}^n$

iv) A has a pivot position in each row of REF of A

$A = \begin{bmatrix} 2 & -1 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}$

columns of A are linearly <u>independent</u>. Columns of A <u>do not</u> span $\mathbb{R}_3$

$\begin{bmatrix} 2 & -1 & -4 \\ 0 & 5 & 7 \\ 0 & 0 & 3 \end{bmatrix}$ → independent also spans $\mathbb{R}_3$

$\begin{bmatrix} 2 & -1 & -4 & 3 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ → Dependend and spans $\mathbb{R}_3$

Decide if the vectors

$$\begin{bmatrix}1\\-1\\0\\2\end{bmatrix} \begin{bmatrix}1\\0\\-2\\0\end{bmatrix} \begin{bmatrix}0\\3\\1\\-1\end{bmatrix} \begin{bmatrix}4\\-1\\1\\3\end{bmatrix}$$ Are linearly dependent.

→ If yes, express one of the vector as a linear combination of other vector

The augmented matrix

$$\left[\begin{array}{cccc|c}1&1&0&4&0\\-1&0&3&-1&0\\0&-2&1&1&0\\2&0&-1&3&0\end{array}\right] \sim \left[\begin{array}{cccc|c}1&1&0&4&0\\0&1&3&3&0\\0&0&1&1&0\\0&0&0&0&0\end{array}\right]$$

Since 4th column does not have a pivot position, c_4 is a free variable

$c_4 = t,\ t \in \mathbb{R}$

$c_3 + c_4 = 0 \Rightarrow c_3 = -t$

$c_1 + c_2 + 4c_4 = 0 \Rightarrow c_1 = -4t$

$c_2 + 3c_3 + 3c_4 = 0 \Rightarrow c_2 = 0$

(General solution in parametric form)

$$\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix} = \begin{bmatrix}-4\\0\\1\\-1\end{bmatrix} t,\ t \in \mathbb{R}.$$ Yes, the given vectors are linearly dependent

(solution)

$$\begin{bmatrix}1\\-1\\0\\1\end{bmatrix} + 0\begin{bmatrix}1\\0\\-2\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\3\\1\\-1\end{bmatrix} + 1\begin{bmatrix}4\\-1\\1\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

A (3x3) E (2x1)

B (3x3)

C (2x2)

D (2x2)

B+A ✓

AC ✗

CD ✓

A+B ✓

CB ✓

EB ✗

(This comes after next page's note)

Find the values(s) of h for which the vectors

$\begin{bmatrix}1\\3\\-3\end{bmatrix} \begin{bmatrix}-2\\-4\\1\end{bmatrix} \begin{bmatrix}-1\\1\\h\end{bmatrix}$ are linearly independent

1) solve $c_1\begin{bmatrix}1\\3\\-3\end{bmatrix} + c_2\begin{bmatrix}-2\\-4\\1\end{bmatrix} + c_3\begin{bmatrix}-1\\1\\h\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $\left[\begin{array}{ccc|c}1 & -2 & -1 & 0\\3 & -4 & 1 & 0\\-3 & 1 & h & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & -2 & -1 & 0\\0 & 1 & 2 & 0\\0 & 0 & 7+h & 0\end{array}\right]$

→ If $7+h=0$, then 3rd column will not have a pivot position
→ If $h=0$ will make the given vectors linearly dependent.

2) solve $c_1\begin{bmatrix}1\\-5\\-2\end{bmatrix} + c_2\begin{bmatrix}-3\\8\\6\end{bmatrix} + c_3\begin{bmatrix}4\\h\\8\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $\left[\begin{array}{ccc|c}1 & -3 & 4 & 0\\-5 & 8 & h & 0\\-2 & 6 & -8 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & -3 & 4 & 0\\0 & -7 & 20+h & 0\\0 & 0 & 0 & 0\end{array}\right]$

Matrix operations

→ An $m \times n$ matrix A is a rectangular array of numbers with m row and n columns

Let $A = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 7\end{bmatrix}_{2\times3}$, $B = \begin{bmatrix}4 & 0\\1 & 3\\8 & 6\end{bmatrix}_{3\times2}$ $C = \begin{bmatrix}3 & 5 & 0\\-1 & 2 & 1\end{bmatrix}_{2\times3}$ $AC \begin{bmatrix}4 & 7 & 3\\3 & 7 & 8\end{bmatrix}_{2\times3}$

$\times$ $+$

→ If A & B are matrices of the same size their sum A+B is a matrix of the same size formed by adding corresponding entries in A & B

→ To multiply two matrices A & B the # of columns A must be the same number of # rows in B

$\underset{2\times3 \cdot 3\times2}{AB} = \underset{2\times2}{AB} \begin{bmatrix}30 & 24\\77 & 57\end{bmatrix}$ $AC_{3\times3} = \begin{bmatrix}7 & 6 & -3\\-4 & -3 & 2\\6 & 6 & -2\end{bmatrix}$ $AC \neq CA$

$A = \begin{bmatrix}2 & 5\\-1 & -3\\2 & 4\end{bmatrix}_{3\times2}$ $\times$ $C = \begin{bmatrix}1 & 3 & 1\\1 & 6 & -1\end{bmatrix}_{3\times2}$ $CA = \begin{bmatrix}1 & 0\\6 & 1\end{bmatrix}$