

CONCORDIA UNIVERSITY

DEPARTMENT OF COMPUTER SCIENCE & SOFTWARE ENGINEERING

COMP 232/4

Mathematics for Computer Science

Winter 2016

Assignment 1 Solutions

1. For each of the following statements use a truth table to determine whether it is a tautology, a contradiction, or a contingency.

$$(a) \underbrace{(p \vee r)}_a \wedge \underbrace{(q \vee r)}_b \leftrightarrow \underbrace{((p \wedge q) \vee r)}_c$$

Solution: Tautology.

| p | q | r | $\underbrace{p \vee r}_a$ | $\underbrace{q \vee r}_b$ | $a \wedge b$ | $p \wedge q$ | $\underbrace{(p \wedge q) \vee r}_c$ | $(a \wedge b) \leftrightarrow c$ |
|-----|-----|-----|---------------------------|---------------------------|--------------|--------------|--------------------------------------|----------------------------------|
| T | T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | T | T | T |
| T | F | T | T | T | T | F | T | T |
| F | T | T | T | T | T | F | T | T |
| T | F | F | T | F | F | F | F | T |
| F | T | F | F | T | F | F | F | T |
| F | F | T | T | T | T | F | T | T |
| F | F | F | F | F | F | F | F | T |

$$(b) \underbrace{(p \wedge (\neg(\neg p \vee q)))}_a \vee \underbrace{(p \wedge q)}_c$$

Solution: Contingency.

| p | q | $\neg p$ | $\neg p \vee q$ | $\underbrace{\neg(\neg p \vee q)}_a$ | $\underbrace{p \wedge a}_b$ | $\underbrace{p \wedge q}_c$ | $b \vee c$ |
|-----|-----|----------|-----------------|--------------------------------------|-----------------------------|-----------------------------|------------|
| T | T | F | T | F | F | T | T |
| T | F | F | F | T | T | F | T |
| F | T | T | T | F | F | F | F |
| F | F | T | T | F | F | F | F |

$$(c) \underbrace{(p \wedge \underbrace{(\neg q \rightarrow \neg p)}_a)}_b \rightarrow q$$

Solution: Tautology.

| p | q | $\neg p$ | $\neg q$ | $\underbrace{\neg q \rightarrow \neg p}_a$ | $\underbrace{p \wedge a}_b$ | $b \rightarrow q$ |
|-----|-----|----------|----------|--|-----------------------------|-------------------|
| T | T | F | F | T | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | F | T |
| F | F | T | T | T | F | T |

$$(d) \underbrace{(p \rightarrow r)}_a \vee \underbrace{(q \rightarrow r)}_b \rightarrow \underbrace{(p \vee q \rightarrow r)}_c$$

Solution: Contingency.

| p | q | r | $\underbrace{p \rightarrow r}_a$ | $\underbrace{q \rightarrow r}_b$ | $a \vee b$ | $p \vee q$ | $\underbrace{(p \vee q \rightarrow r)}_c$ | $(a \vee b) \rightarrow c$ |
|-----|-----|-----|----------------------------------|----------------------------------|------------|------------|---|----------------------------|
| T | T | T | T | T | T | T | T | T |
| T | T | F | F | F | F | T | F | T |
| T | F | T | T | T | T | T | T | T |
| F | T | T | T | T | T | T | T | T |
| T | F | F | F | T | T | T | F | F |
| F | T | F | T | F | T | T | F | F |
| F | F | T | T | T | T | F | T | T |
| F | F | F | T | T | T | F | T | T |

2. For each of the following logical equivalences state whether it is valid or invalid. If invalid then give a counterexample (*e.g.*, based on a truth table). If valid then give an algebraic proof using logical equivalences from Tables 6, 7, and 8 from Section 1.3 of textbook.

(a) $p \rightarrow (q \rightarrow r) \equiv q \rightarrow (\neg p \vee r)$

Solution: Valid.

$$\begin{aligned}
 & p \rightarrow (q \rightarrow r) \\
 \equiv & \neg p \vee (q \rightarrow r) && \text{law for conditional} \\
 \equiv & \neg p \vee (\neg q \vee r) && \text{law for conditional} \\
 \equiv & (\neg p \vee \neg q) \vee r && \text{associativity} \\
 \equiv & (\neg q \vee \neg p) \vee r && \text{commutativity} \\
 \equiv & \neg q \vee (\neg p \vee r) && \text{associativity} \\
 \equiv & q \rightarrow (\neg p \vee r) && \text{law for conditional}
 \end{aligned}$$

(b) $(p \rightarrow r) \wedge (q \rightarrow r) \equiv ((p \wedge q) \rightarrow r)$

Solution: Invalid.

If $p = T$, $q = F$, and $r = F$ then the LHS is False, while the RHS is True.

(c) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv (p \rightarrow (q \wedge r))$

Solution: Valid.

$$\begin{aligned}
 & (p \rightarrow q) \wedge (p \rightarrow r) \\
 \equiv & (\neg p \vee q) \wedge (\neg p \vee r) && \text{law for conditional} \\
 \equiv & \neg p \vee (q \wedge r) && \text{distributivity} \\
 \equiv & (p \rightarrow (q \wedge r)) && \text{law for conditional}
 \end{aligned}$$

(d) $((p \vee q) \wedge (\neg p \vee r)) \equiv (q \vee r)$

Solution: Invalid.

If $p = T$, $q = T$, and $r = F$ then the LHS is False, while the RHS is True.

3. Write down the negations of each of the following statements *in their simplest form* (i.e., do not simply state “It is not the case that...”). Below, x denotes a real number, $x \in \mathbf{R}$.

(a) The plane is early or my watch is slow.

Solution: *The plane is on time and my watch is on time.*

This is of the form $e \vee s$. The negation is $\neg(e \vee s) \equiv \neg e \wedge \neg s$.

(b) Doing the assignments is a sufficient condition for John to pass the course.

Solution: *John does the assignments but does not pass the course.*

This is of the form $a \rightarrow p$. The negation is $\neg(a \rightarrow p) \equiv a \wedge \neg p$.

(c) If x is positive, then x is not negative and x is not 0.

Solution: *x is positive, and x is negative or x is 0.*

This is of the form $p \rightarrow (\neg n \wedge \neg z)$.

The negation is $\neg(p \rightarrow (\neg n \wedge \neg z)) \equiv \neg p \vee \neg(\neg n \wedge \neg z) \equiv \neg p \vee n \vee z$.

(d) $(0 < x \leq 1) \vee (-1 < x < 0)$

Solution: $(x = 0) \vee (x \leq 1) \vee (x > 1)$.

The negation is that x does not lie in the half-open interval $(0, 1]$ and x does not lie in the open interval $(-1, 0)$

4. Write the following statements in predicate form, using logical operators \wedge , \vee , \neg , and quantifiers \forall , \exists . Below \mathbb{Z}^+ denotes all positive integers $\{1, 2, 3, \dots\}$.

We assume that the Universe of Discourse is all numbers (natural \mathbb{N} , integers \mathbb{Z} , rational \mathbb{Q} , irrational \mathbb{R} , complex \mathbb{C} , ...).

- (a) The square of a positive integer is always bigger than the integer.

Solution: $\forall x \left(x \in \mathbb{Z}^+ \rightarrow x^2 > x \right)$

- (b) There is no integer solution to the equation $x = x + 1$.

Solution: $\forall x \left(x \in \mathbb{Z} \rightarrow x \neq x + 1 \right)$ or $\neg \left[\exists x \left(x \in \mathbb{Z} \wedge x = x + 1 \right) \right]$

- (c) The absolute value of an integer is not necessarily positive.

Solution: $\exists x \left(x \in \mathbb{Z} \wedge |x| \not> 0 \right)$

- (d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of those integers.

Solution: $\forall x \forall y \left((x \in \mathbb{Z} \wedge y \in \mathbb{Z}) \rightarrow |x + y| \leq |x| + |y| \right)$

5. Let P and Q be predicates on the set S , where S has two elements, say, $S = \{a, b\}$. Then the statement $\forall xP(x)$ can also be written in full detail as $P(a) \wedge P(b)$. Rewrite each of the statements below in a similar fashion, using P , Q , and logical operators, but without using quantifiers.

$$(a) \quad \forall x, y(P(x) \vee Q(y)) \equiv \forall x(\forall y(P(x) \vee Q(y)))$$

Solution:

$$\begin{aligned} & \forall x(\forall y(P(x) \vee Q(y))) \\ \equiv & \quad \forall y(P(a) \vee Q(y)) \wedge \forall y(P(b) \vee Q(y)) \\ \equiv & \quad \left[(P(a) \vee Q(a)) \wedge (P(a) \vee Q(b)) \right] \\ & \wedge \left[(P(b) \vee Q(a)) \wedge (P(b) \vee Q(b)) \right] \\ & \wedge \left[(P(c) \vee Q(a)) \wedge (P(c) \vee Q(b)) \right] \\ \equiv & \quad \left[P(a) \vee (Q(a) \wedge Q(b)) \right] \\ & \wedge \left[P(b) \vee (Q(a) \wedge Q(b)) \right] \\ & \wedge \left[P(c) \vee (Q(a) \wedge Q(b)) \right] \end{aligned}$$

$$(b) \quad \exists xP(x) \vee \exists xQ(x)$$

Solution:

$$\begin{aligned} & \exists xP(x) \vee \exists xQ(x) \\ \equiv & \quad (P(a) \vee P(b)) \vee (\exists xQ(x)) \\ \equiv & \quad (P(a) \vee P(b)) \vee (Q(a) \vee Q(b)) \end{aligned}$$

$$(c) \quad \exists xP(x) \wedge \exists xQ(x)$$

Solution:

$$\begin{aligned} & \exists xP(x) \wedge \exists xQ(x) \\ \equiv & \quad (P(a) \vee P(b)) \wedge (\exists xQ(x)) \\ \equiv & \quad (P(a) \vee P(b)) \wedge (Q(a) \vee Q(b)) \end{aligned}$$

$$(d) \exists x, y(P(x) \wedge Q(y)) \equiv \exists x(\exists y(P(x) \wedge Q(y)))$$

Solution:

$$\begin{aligned} & \exists x(\exists y(P(x) \wedge Q(y))) \\ \equiv & \exists y(P(a) \wedge Q(y)) \vee \exists y(P(b) \wedge Q(y)) \\ \equiv & \left[(P(a) \wedge Q(a)) \vee (P(a) \wedge Q(b)) \right] \\ & \vee \left[(P(b) \wedge Q(a)) \vee (P(b) \wedge Q(b)) \right] \\ \equiv & \left[P(a) \wedge (Q(a) \vee Q(b)) \right] \\ & \vee \left[P(b) \wedge (Q(a) \vee Q(b)) \right] \end{aligned}$$

$$(e) \forall x \exists y(P(x) \wedge Q(y)) \equiv \forall x(\exists y(P(x) \wedge Q(y)))$$

Solution:

$$\begin{aligned} & \forall x(\exists y(P(x) \wedge Q(y))) \\ \equiv & \exists y(P(a) \wedge Q(y)) \wedge \exists y(P(b) \wedge Q(y)) \\ \equiv & \left[(P(a) \wedge Q(a)) \vee (P(a) \wedge Q(b)) \right] \\ & \wedge \left[(P(b) \wedge Q(a)) \vee (P(b) \wedge Q(b)) \right] \\ \equiv & \left[P(a) \wedge (Q(a) \vee Q(b)) \right] \\ & \wedge \left[P(b) \wedge (Q(a) \vee Q(b)) \right] \\ & \wedge \left[P(c) \wedge (Q(a) \vee Q(b)) \right] \end{aligned}$$

6. Let the domain for x be the set of all students in this class and the domain for y be the set of all countries in the world. Let $P(x, y)$ denote student x has visited country y and $Q(x, y)$ denote student x has a friend in country y . Express each of the following using logical operations and quantifiers, and the propositional functions $P(x, y)$ and $Q(x, y)$.

(a) Carlos has visited Bulgaria.

Solution: $P(\text{Carlos}, \text{Bulgaria})$

(b) Every student in this class has visited the United States.

Solution: $\forall x P(x, \text{UnitedStates})$

(c) Every student in this class has visited some country in the world.

Solution: $\forall x \exists y P(x, y)$

(d) There is no country that every student in this class has visited.

Solution: $\forall y \exists x \neg P(x, y)$

(e) There are two students in this class, who between them, have a friend in every country in the world.

Solution: $\exists x \exists y \left(x \neq y \wedge \forall z [Q(x, z) \vee Q(y, z)] \right)$

7. For each part in the previous question, form the negation of the statement so that all negation symbols occur immediately in front of predicates. For example:

$$\neg(\forall x(P(x) \wedge Q(x))) \equiv \exists x(\neg((P(x) \wedge Q(x)))) \equiv \exists x((\neg P(x)) \vee (\neg Q(x)))$$

- (a) **Solution:** $\neg P(\text{Carlos}, \text{Bulgaria})$

Carlos has not visited Bulgaria

- (b) **Solution:** $\neg(\forall x P(x, \text{UnitedStates})) \equiv \exists x(\neg P(x, \text{UnitedStates}))$

There is a student in this class who has not visited the United States

- (c) **Solution:** $\neg(\forall x [\exists y P(x, y)]) \equiv \exists x \neg[\exists y P(x, y)] \equiv \exists x \forall y [\neg P(x, y)]$

There is a student in this class who has not visited any country

- (d) **Solution:**

$$\begin{aligned} \neg(\forall y [\exists x \neg P(x, y)]) &\equiv \\ \exists y \neg[\exists x \neg P(x, y)] &\equiv \\ \exists y \forall x \neg[\neg P(x, y)] &\equiv \\ \exists y \forall x [\neg\neg P(x, y)] &\equiv \\ \exists y \forall x P(x, y) & \end{aligned}$$

There is a country that every student in this class has visited

- (e) **Solution:**

$$\begin{aligned} \neg[\exists x (\exists y (x \neq y \wedge \forall z [Q(x, z) \vee Q(y, z)]))] &\equiv \\ \forall x [\neg(\exists y (x \neq y \wedge \forall z [Q(x, z) \vee Q(y, z)]))] &\equiv \\ \forall x [\forall y \neg(x \neq y \wedge \forall z [Q(x, z) \vee Q(y, z)])] &\equiv \\ \forall x [\forall y (x = y \vee \neg(\forall z [Q(x, z) \vee Q(y, z)]))] &\equiv \\ \forall x [\forall y (x = y \vee \exists z (\neg[Q(x, z) \vee Q(y, z)]))] &\equiv \\ \forall x [\forall y (x = y \vee \exists z (\neg Q(x, z) \wedge \neg Q(y, z)))] & \end{aligned}$$

For every pair of distinct students in this class, there is a country where neither one of them has a friend

8. Negate the following statements and transform the negation so that negation symbols immediately precede predicates. (See example in Question 7.)

$$(a) \exists x \exists y (P(x, y)) \vee \forall x \forall y (Q(x, y))$$

Solution:

$$\begin{aligned} & \neg (\exists x \exists y P(x, y) \vee \forall x \forall y Q(x, y)) \\ \equiv & \neg (\exists x \exists y P(x, y)) \wedge \neg (\forall x \forall y Q(x, y)) \\ \equiv & (\forall x \forall y \neg P(x, y)) \wedge (\exists x \exists y \neg Q(x, y)) \end{aligned}$$

$$(b) \forall x \forall y (Q(x, y) \leftrightarrow Q(y, x))$$

Solution:

$$\begin{aligned} & \neg (\forall x \forall y (Q(x, y) \leftrightarrow Q(y, x))) \\ \equiv & \exists x \exists y (\neg (Q(x, y) \leftrightarrow Q(y, x))) \\ \equiv & \exists x \exists y \neg ((Q(x, y) \rightarrow Q(y, x)) \wedge (Q(y, x) \rightarrow Q(x, y))) \\ \equiv & \exists x \exists y (\neg (Q(x, y) \rightarrow Q(y, x)) \vee \neg (Q(y, x) \rightarrow Q(x, y))) \\ \equiv & \exists x \exists y (\neg (\neg Q(x, y) \vee Q(y, x)) \vee \neg (\neg Q(y, x) \vee Q(x, y))) \\ \equiv & \exists x \exists y ((Q(x, y) \wedge \neg Q(y, x)) \vee (Q(y, x) \wedge \neg Q(x, y))) \end{aligned}$$

$$(c) \forall y \exists x \exists z (T(x, y, z) \wedge Q(x, y))$$

Solution:

$$\begin{aligned} & \neg (\forall y \exists x \exists z (T(x, y, z) \wedge Q(x, y))) \\ \equiv & \exists y \forall x \forall z \neg (T(x, y, z) \wedge Q(x, y)) \\ \equiv & \exists y \forall x \forall z (\neg T(x, y, z) \vee \neg Q(x, y)) \end{aligned}$$