Propositional Equivalences

A **tautology** is a compound proposition that is *True* for all truth values of the atomic propositions in it. Example:

$$p \vee \neg p$$

A **contradiction** is a compound proposition that is *False* for all truth values of the atomic propositions in it. Example:

$$p \wedge \neg p$$

A **contingency** is a proposition which is neither a tautology nor a contradiction. Example:

$$p \vee q$$

Definition:

Two compound propositions p, q are **logically equivalent** if $p \longleftrightarrow q$ is a tautology. This is denoted

$$p \equiv q$$

Alternative definition:

Two compound propositions p, q are **logically equivalent** if they have the same truth table.

- Example:

Is the proposition $p \rightarrow q$ logically equivalent to the proposition $\neg p \vee q$?

p	q	$\neg p$	$\overbrace{\neg p \vee q}^{a}$	$\overbrace{p \rightarrow q}^{b}$	$a \leftrightarrow b$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Since the last column is all *True*, we have

$$p \rightarrow q \equiv \neg p \vee q.$$

- Example:

Are the compound propositions $(p \to q) \wedge (q \to p)$ and $p \leftrightarrow q$ logically equivalent?

p	q	$\overbrace{p \to q}^{a}$	$\overbrace{q \to p}^{b}$	$a \wedge b$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Since the two righ-most columns are the same:

$$(p \to q) \wedge (q \to p) \quad \equiv \quad p \leftrightarrow q$$

- Example: Show that

$$(p \wedge q) \vee (\neg p \wedge \neg q) \;\equiv\; p \leftrightarrow q.$$

p	q	$\overbrace{p \wedge q}^{a}$	$\neg p$	$\neg q$	$\overbrace{\neg p \wedge \neg q}^{b}$	$a \vee b$	$p \leftrightarrow q$
T	T	T	F	F	F	T	T
T	F	F	F	T	F	F	F
F	T	F	T	F	F	F	F
F	F	F	T	T	T	T	T

- Example: Using truth tables, determine whether the following proposition is a tautology, contradiction or a contingency.

$$\big((p \to q) \to r\big) \leftrightarrow \big((p \to q) \land (p \to r)\big).$$

- Example:

$$\Big(\big(\neg(q \to p)\big) \land \neg r\Big) \stackrel{?}{\equiv} \Big(\neg p \lor (\neg q \lor r)\Big)$$

Basic logical equivalences:

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotency
$\neg(\neg p) \equiv p$	Double negation
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutativity
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associativity
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributivity
$\neg(p \wedge q) \equiv (\neg p \vee \neg q)$ $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$	de Morgan's laws

Other useful equivalences

$$\text{Negation laws}: \begin{cases} (\neg p) \vee p \equiv T \\ (\neg p) \wedge p \equiv F \end{cases}$$

$$\text{Absorption laws}: \begin{cases} p \wedge (p \vee q) \equiv p \\ p \vee (p \wedge q) \equiv p \end{cases}$$

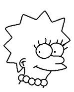
A proof of the last equivalence:

$$\begin{array}{lll} & p \vee (p \wedge q) & \\ \equiv & (p \wedge T) \vee (p \wedge q) & \text{identity} \\ \equiv & p \wedge (T \vee q) & \text{distributivity} \\ \equiv & p \wedge T & \text{domination} \\ \equiv & p & \text{identity} \end{array}$$

Use brackets to avoid ambiguity!

Contrapositive, converse and inverse

Consider the proposition $p \to q$.

 $q \to p$ is the **converse** of $p \to q$

 $\neg q \to \neg\ p$ is the **contrapositive** of $p \to q$

 $\neg p \to \neg q$ is the **inverse** of $p \to q$

- Example: $p \rightarrow q$

"If you are a Computer Science student, you can take COMP 238."

Converse: $q \rightarrow p$

"If you can take COMP 238, you are a Computer Science student."

Contrapositive: $\neg q \rightarrow \neg p$

"If you cannot take COMP 238, you are not a Computer Science student."

Inverse: $\neg p \rightarrow \neg q$

"If you are not a Computer Science student, you cannot take COMP 238."

Contrapositive, converse and inverse ...

The proposition $p \to q$ is logically equivalent to its contrapositive $\neg q \to \neg p$.

Proof:

$$
\begin{array}{lll}
 & p \to q & \\
\equiv & \neg p \vee q & \text{Example above} \\
\equiv & q \vee \neg p & \text{Commutativity} \\
\equiv & \neg(\neg q) \vee \neg p & \text{Double negation} \\
\equiv & \neg q \to \neg p. & \text{Example above}
\end{array}
$$

However, neither the inverse nor the converse of $p \to q$ is logically equivalent to it.

Everyone should know:

the definition of a

proposition, tautology, contradiction, contingency,

all **logical operations**,

basic **logical equivalences**,

how to **construct** a **truth table**,

how to **use** logical equivalences.