Discrete Mathematics

- Discrete mathematics is devoted to the study of **discrete** or distinct unconnected objects.

- Classical mathematics deals with functions on real numbers. Real numbers form a continuous line. Some calculus techniques apply only to continuous functions.

- Dealing with discrete objects requires techniques different from classical math.

Why study Discrete Mathematics?

Computers typically work with discrete information.

•Examples:

bits
integers
letters
employee records
passwords

This is why a course in Discrete Mathematics is standard in Computer Science or Software Engineering programmes.

Applications of Discrete Mathematics

The techniques of Discrete Mathematics help us solve many kinds of problems. For example:

- What is the shortest route to go from point A to point B given a map marked with all distances between points?
- How many different ways are there of choosing a valid password in a system?
- What is the most efficient way to multiply a given sequence of matrices?
- How should you schedule a given collection of tasks on a set of computers, so that all tasks finish as soon as possible?

Applications of Discrete Mathematics

Discrete mathematics provides the foundations for many fields:

- Computer security and cryptography.
- Automata theory: the theory behind compilers.
- Algorithms and data structures.
- Database theory.
- Routing and other problems in computer networks.
- Scheduling theory.

Learning objectives for discrete mathematics

- To develop **mathematical maturity**. IOW, the ability to understand and create mathematical arguments.

- To learn **mathematical reasoning** and **problem solving**, rather than a set of skills (e.g. how to multiply, divide, differentiate and integrate).

- To be able to attack problems **different from previously seen**.

- To learn to apply **mathematical abstraction** to problems.

- To understand that computer programming is **not that different from** writing mathematical proofs.

HOW to study Discrete Math?

Goal: Develop your ability to understand and create mathematical arguments, i.e. to develop **mathematical maturity**. Discrete Math is **different** from the Math courses you have studied before:

1. **Not** a set of formulas you need to memorize and be able to apply.
2. You will learn Mathematical **reasoning** and **problem solving**. Mathematical reasoning and problem solving are essential skill in writing computer programs and developing software.

In order to achieve this:

1. **Read** the textbook, not only the lecture slides.
2. Solve exercises, Solve exercises, Solve exercises, ...
3. Use the textbooks **Student site**
4. Practise, practise, practise, ...

Syllabus

In this course we cover

1. Propositional logic, first order logic, proofs.

2. Sets, functions, sequences, and sums.

3. Integers and modular arithmetic.

4. Induction and recursion.

5. Relations, equivalence relations, partial orderings.

Propositional Logic

Logic deals with the methods of reasoning. The rules of logic give precise meaning to mathematical statements.

It deals with objects having two values:

• *True*, also denoted T or $\top$ or 1

• *False*, also denoted F or $\bot$ or 0

We call T and F **truth values**.

Proposition or Statement: A declarative sentence which is either true or false but not both. (We say its *truth value* is either T or F.)

Examples of propositions

"*Montreal is a city in Canada.*"

- truth value is T.

"*Concordia is located near a metro station*"

- truth value is T.

"$2 + 2 = 5$"

- truth value is F.

Examples of things that are **not** propositions

"*Don't do that!*"

- an *imperative* sentence.

"*What time is it?*"

- A *question*

"$x < 4$"

- truth value depends on x.

We name propositions, for example *p, q, r, . . .*

•Examples:

$p =_{\text{def}}$ *"It is raining today."*

$q =_{\text{def}}$ *"Montreal is the capital of Canada."*

$r =_{\text{def}}$ "$2 + 3 = 5$."

•The truth value of some proposition dependes on their time, place, context, speaker, ...

"Bill Clinton is the president of the USA"

"The sun is shining"

•The truth value of other propositions are absolute, that is, either True or False

So, the only way to interpret "$2+2 = 5$" is *False*.

and the only way to interpret "$2+2 = 4$" is *True*.

Logical operators and compound propositions

Propositions can be combined using logical connectives, such as **negation, and, or**, etc.

•Examples:

$\neg p$	"*not* p"
$p \wedge q$	"p *and* q"
$p \vee q$	"p *or* q"
$p \oplus q$	"p *exclusive-or* q"
$p \rightarrow q$	"*if* p *then* q"
$p \leftrightarrow q$	"p *if and only if* q"

Negation

"It is not the case that p."

$$\neg p$$

If the proposition p is true then the negation of p is false.

If the proposition p is false then the negation of p is true.

•Example:

$p =_{\text{def}}$ *"Today is Monday."*

$\neg p =$ *"It is not the case that today is Monday."*

"Today is not Monday."

Truth tables

We use a **truth table** to determine the truth values of compound propositions in terms of the component parts.

- The truth table for negation:

p	$\neg p$
T	F
F	T

Disjunction

$$p \vee q$$

is true when at least one of p, q is true
(also known as *inclusive or*).

•Example:

$p =_{\text{def}}$ "*Montreal is the capital of Canada.*"

$q =_{\text{def}}$ "$2 + 3 = 5$."

$p \vee q =$
"*Montreal is the capital of Canada or* $2+3 = 5$."

•The truth table for disjunction:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conjunction

$$p \wedge q$$

is true when both p, q are true.

•Example:

$p =_{\text{def}}$ "*It is raining.*"

$q =_{\text{def}}$ "*It is dark outside.*"

$p \wedge q =$ "*It is raining and it is dark outside.*"

•The truth table for conjunction:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Exclusive Or

$$p \oplus q$$

is true when exactly one of p, q is true and the other is false.

•Example:

$p =_{\text{def}}$ *"I will have soup."*

$q =_{\text{def}}$ *"I will have salad."*

$p \oplus q =$ *"I will have soup or salad but not both."*

•The truth table for exclusive or:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional

$$p \rightarrow q$$
(if p then q)

$p \rightarrow q$ is false only when p is true and q is false.

p is called the **hypothesis** or **antecedent**

q is called the **conclusion** or **consequent**.

•Example: $p \rightarrow q$ $=_{\text{def}}$

"If today is Monday, then I have to go to school."

•The truth table for conditional:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The conditional $p \to q$ can be expressed in English in many ways:

if p then q

p implies q

p only if q

p is sufficient for q

q is necessary for p

q if p

q whenever p

Examples:

- *"If you get 100% on the final, then you will get an A."*

- *"If Maria learns discrete mathematics, then she will find a good job."*

  - *"Maria will find a good job when she learns discrete mathematics."*

  - *"For Maria to get a good job, it is sufficient for her to learn discrete mathematics.'*

•Caution is required:

"If today is Friday, then $2+3=5$."
"If today is Friday, then $2+3=6$."

"If it is sunny today, then we will go to the beach."

Biconditional

$p \leftrightarrow q$ (p if and only if q) is true when p, q have the same truth values.

p is necessary and sufficient for q.

p iff q.

•Example:

$p \leftrightarrow q$ $=_{\text{def}}$

"*We will go to the beach if and only if it is sunny.*"

•The truth table for biconditional:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Classroom exercise:

Build a truth table for the compound proposition

$$\neg((p \wedge q) \vee r)$$