MAT 2384 3X Assignment 2 : Solutions

1. $y' + \frac{3}{x} y = 12x^2, \quad y(1) = 5$

this is first-order linear with $f(x) = \frac{3}{x}$ and $r(x) = 12x^2$

then $\mu(x) = e^{\int f(x)dx} = e^{\int \frac{3}{x} dx} = e^{3\ln x} = x^3$

and $y(x) = \dfrac{\int \mu(x) r(x) dx + C}{\mu(x)} = x^{-3}\left[\int (x^3)(12x^2)dx + C\right]$

$= x^{-3}\left[\int 12x^5 dx + C\right]$

$= x^{-3}(2x^6 + C)$

(general solution) $= 2x^3 + Cx^{-3}$

then $y(1) = 5 \Rightarrow 5 = 2(1)^3 + C(1)^{-3} \Rightarrow C = 3$

$\therefore$ the unique solution is $\boxed{y(x) = 2x^3 + 3x^{-3}}$

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2. $y' - 4y = 7e^{2x}, \quad y(0) = 0$

first-order linear with $f(x) = -4$ and $r(x) = 7e^{2x}$

so $\mu(x) = e^{\int -4dx} = e^{-4x}$

and $y(x) = e^{4x}\left[\int (e^{-4x})(7e^{2x})dx + C\right]$

$= e^{4x}\left(\int 7e^{-2x} dx + C\right)$

$= e^{4x}\left(-\frac{7}{2}e^{-2x} + C\right) = -\frac{7}{2}e^{2x} + Ce^{4x}$ (general solution) then $y(0)=0 \Rightarrow 0 = -\frac{7}{2}e^0 + Ce^0 \Rightarrow C = 7/2$

thus the <u>unique solution</u> is $\boxed{y(x) = \frac{7}{2}\left(e^{4x} - e^{2x}\right)}$

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3. $y' + y\tan x = y^2, \quad y(0) = 1/4$

this is a Bernoulli equation with $p(x) = \tan x$, $q(x) = 1$ and $a = 2$
let $u(x) = (y(x))^{1-a} = (y(x))^{1-2} = (y(x))^{-1}$ and then the

DE becomes $u' + (1-a)p(x)u = (1-a)q(x)$ or $u' - u\tan x = -1$

then $\mu(x) = e^{\int -\tan x\,dx} = e^{\int(-\sin x/\cos x)dx} = e^{\ln\cos x} = \cos x$

so
$$\begin{aligned} u(x) &= (\cos x)^{-1}\left[\int (\cos x)(-1)\,dx + C\right] \\ &= \sec x\left(\int -\cos x\,dx + C\right) \\ &= \sec x\,(-\sin x + C) \\ &= C\sec x - \tan x \end{aligned}$$

then the <u>general solution</u> is $y(x) = (u(x))^{-1} = \dfrac{1}{C\sec x - \tan x}$

$y(0) = 1/4 \Rightarrow \dfrac{1}{4} = \dfrac{1}{C\sec(0) - \tan(0)} \Rightarrow C = 4$

and the <u>unique solution</u> is $\boxed{y(x) = \dfrac{1}{4\sec x - \tan x}}$

4. $y'' - 2\sqrt{2}y' + 2y = 0$, $y(0)=0$, $y'(0)=\sqrt{2}$

the characteristic equation is $\lambda^2 - 2\sqrt{2}\lambda + 2 = (\lambda - \sqrt{2})^2$, so the roots are $\lambda_1 = \lambda_2 = \sqrt{2}$ and the <u>general solution</u> is
$y(x) = c_1 e^{\sqrt{2}x} + c_2 x e^{\sqrt{2}x}$

$y(0) = 0 \Rightarrow c_1 e^0 + c_2 (0) e^0 = 0 \Rightarrow c_1 = 0$
$y'(x) = \sqrt{2}\, c_1 e^{\sqrt{2}x} + c_2 e^{\sqrt{2}x} + \sqrt{2}\, c_2 x e^{\sqrt{2}x}$
$y'(0) = \sqrt{2} \Rightarrow \sqrt{2}\, c_1 e^0 + c_2 e^0 + \sqrt{2}\, c_2 (0) e^0 = \sqrt{2} \Rightarrow c_2 = \sqrt{2}$

$\therefore$ the <u>unique solution</u> is $\boxed{y(x) = \sqrt{2}\, x e^{\sqrt{2}x}}$

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5. $y'' + 3y' - 28y = 0$, $y(0) = 3$, $y'(0) = 1$

the char. eq. is $\lambda^2 + 3\lambda - 28 = (\lambda+7)(\lambda-4) = 0$, so the roots are $\lambda_1 = -7$ and $\lambda_2 = 4$ and the <u>general solution</u> is
$y(x) = c_1 e^{-7x} + c_2 e^{4x}$

$y(0) = 3 \Rightarrow c_1 e^0 + c_2 e^0 = 3 \Rightarrow c_1 + c_2 = 3$
$y'(x) = -7c_1 e^{-7x} + 4c_2 e^{4x}$
$y'(0) = 1 \Rightarrow -7c_1 e^0 + 4c_2 e^0 = 1 \Rightarrow -7c_1 + 4c_2 = 1$

$\left.\begin{matrix} c_1 + c_2 = 3 \\ -7c_1 + 4c_2 = 1 \end{matrix}\right\} \begin{matrix} c_1 = 1 \\ c_2 = 2 \end{matrix}$

and the <u>unique solution</u> is $\boxed{y(x) = e^{-7x} + 2e^{4x}}$

3X11
A2
4

6. $y'' - 6y' + 13y = 0, \quad y(0) = 1, \; y'(0) = -3$

the char. eq. is $\lambda^2 - 6\lambda + 13 = 0$, so the roots are

$$\lambda_{1,2} = \frac{6 \pm \sqrt{(6)^2 - 4(1)(13)}}{2(1)} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$$

and the general solution is $y(x) = c_1 e^{3x}\cos(2x) + c_2 e^{3x}\sin(2x)$

$y(0) = 1 \Rightarrow c_1 e^0 \cos(0) + c_2 e^0 \sin(0) = 1 \Rightarrow c_1 = 1$

$y'(x) = 3c_1 e^{3x}\cos(2x) - 2c_1 e^{3x}\sin(2x) + 3c_2 e^{3x}\sin(2x) + 2c_2 e^{3x}\cos(2x)$

$y'(0) = -3 \Rightarrow 3c_1 e^0\cos(0) - 2c_1 e^0 \sin(0) + 3c_2 e^0 \sin(0) + 2c_2 e^0 \cos(0)$

$= 3c_1 + 2c_2 = -3 \Rightarrow c_2 = -3$

so the unique solution is $\boxed{y(x) = e^{3x}\cos(2x) - 3e^{3x}\sin(2x)}$

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7. $f(x) = x^3 + 8x - 7 \Rightarrow f'(x) = 3x^2 + 8$

Newton's Method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^3 + 8x_n - 7}{3x_n^2 + 8}$

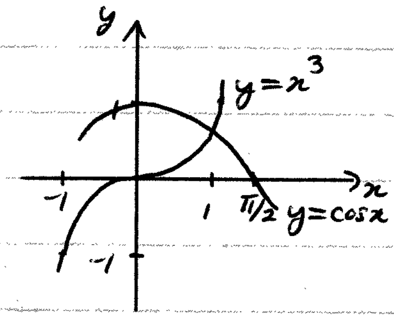
$$= \frac{2x_n^3 + 7}{3x_n^2 + 8}$$

$x_0 = 0.75, \quad x_1 = \dfrac{2(0.75)^3 + 7}{3(0.75)^2 + 8} = 0.80968$

$x_2 = \dfrac{2(0.80968)^3 + 7}{3(0.80968)^2 + 8} = 0.80885$

$x_3 = \dfrac{2(0.80885)^3 + 7}{3(0.80885)^2 + 8} = 0.80885 \quad \therefore$ stop

$\therefore$ the root is $\underline{0.80885}$ (which we knew from assignment #1)

8. 

let $f(x) = x^3 - \cos x$

then $f'(x) = 3x^2 + \sin x$

so $x_{n+1} = x_n - \dfrac{x_n^3 - \cos(x_n)}{3x_n^2 + \sin(x_n)}$

$x_0 = 1$, $x_1 = 1 - \dfrac{(1)^3 - \cos(1)}{3(1)^2 + \sin(1)} = 0.880\,333$

$x_2 = 0.880\,333 - \dfrac{(0.880\,333)^3 - \cos(0.880333)}{3(0.880\,333)^2 + \sin(0.880\,333)}$

$= 0.865\,684$

$x_3 = 0.865\,684 - \dfrac{(0.865684)^3 - \cos(0.865684)}{3(0.865684)^2 + \sin(0.865684)}$

$= 0.865\,474$

$x_4 = 0.865\,474 - \dfrac{(0.865474)^3 - \cos(0.865474)}{3(0.865474)^2 + \sin(0.865474)}$

$= 0.865\,474$ $\therefore$ stop

$\therefore$ the solution is $\underline{x = 0.865\,474}$

okay!

(Check : $(0.865474)^3 = 0.648\,279$, $\cos(0.865474) = 0.648279$)