MAT 2384 A Assignment #1: Solutions

1. $(1+x)y^2y' = 2, \quad y(0) = 0$

the DE is separable: $y^2y' = \frac{2}{1+x}$

or $y^2\,dy = \frac{2}{1+x}\,dx$

integrate on both sides $\int y^2\,dy = \int \frac{2}{1+x}\,dx + C$

to get $\frac{1}{3}y^3 = 2\ln|1+x| + C$

or $y = (6\ln|1+x| + C)^{1/3}$ (general solution)

then $y(0)=0 \Rightarrow 0 = (6\ln(1) + C)^{1/3} \Rightarrow C = 0$

$\therefore$ the unique solution is $\boxed{y = (6\ln|1+x|)^{1/3}}$

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2. $\cot x \sin y\, y' = 1, \quad y(0) = 0$

the DE is separable: $\sin y\, y' = \tan x$

or $\sin y\,dy = \tan x\,dx$

integrate on both sides: $\int \sin y\,dy = \int \tan x\,dx + C$

we get $-\cos y = -\ln|\cos x| + C$

or $\cos y = \ln|\cos x| + C$

and so the general solution is $y = \arccos(\ln|\cos x| + C)$

$y(0) = 0 \Rightarrow 0 = \arccos(\ln(1) + C) = \arccos(C)$

then $C = 1$

$\therefore$ the unique solution is $\boxed{y = \arccos(\ln|\cos x| + 1)}$

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3. $xy' = y + x\sec(y/x)\,, \quad y(1) = \pi/2$

rewrite as $(y + x\sec(y/x))dx - x\,dy = 0$

then $M(x,y) = y + x\sec(y/x)$ } both homogeneous of degree 1
$N(x,y) = -x$

so let $u = y/x$ or $y = ux$ and $dy = u\,dx + x\,du$
and the DE becomes

$$(ux + x\sec u)dx - x(u\,dx + x\,du) = 0$$

or $ux\,dx + x\sec u\,dx - xu\,dx - x^2 du = 0$

or $x\sec u\,dx - x^2\,du = 0$

or $\cos u\,du = \frac{1}{x}dx$ (separable)

integrate $\int \cos u\,du = \int \frac{1}{x}\,dx + C$

to get $\sin u = \ln|x| + C$

then $u = \arcsin(\ln|x| + C)$

but $u = y/x$, so $y = x\arcsin(\ln|x| + C)$ (general solution)

but $y(1) = \pi/2 \Rightarrow \pi/2 = (1)\arcsin(\ln(1) + C) = \arcsin C \Rightarrow C = 1$

$\therefore$ the unique solution is $\boxed{y = x\arcsin(1 + \ln|x|)}$

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4. $((x+1)e^x - e^y)dx - xe^y dy = 0\,, \quad y(1) = 0$

$M(x,y) = (x+1)e^x - e^y \qquad M_y = -e^y$
$N(x,y) = -xe^y \qquad N_x = -e^y$ } $M_y = N_x$ so DE is exact

then $F(x,y) = \int M(x,y)\,dx + g(y)$ (or $\int N(x,y)\,dy + h(x)$)

$= \int((x+1)e^x - e^y)\,dx + g(y)$

$= xe^x - xe^y + g(y)$

but $\frac{\partial F}{\partial y} = -xe^y + g'(y) = N(x,y) = -xe^y \Rightarrow g'(y) = 0$

so we take $g(y)=0$ and $F(x,y) = x(e^x - e^y)$

the general solution is $x(e^x - e^y) = C$

$y(1)=0 \Rightarrow (1)(e^1 - e^0) = C \Rightarrow C = e-1$

$\therefore$ the unique solution is $\boxed{x(e^x - e^y) = e-1}$

or $\boxed{y = \ln\left(e^x - \frac{e-1}{x}\right)}$

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5. $(4xy^4 + 3y^2)dx + (8x^2y^3 + 6xy + 3y^2)dy = 0, \quad y(1) = -1$

$M(x,y) = 4xy^4 + 3y^2 \Rightarrow M_y = 16xy^3 + 6y$
$N(x,y) = 8x^2y^3 + 6xy + 3y^2 \Rightarrow N_x = 16xy^3 + 6y$
$\Big\}\ M_y = N_x$ DE is exact

then $F(x,y) = \int N(x,y)\,dy + g(x)$ (or $\int M(x,y)\,dx + g(y)$)
$= \int (8x^2y^3 + 6xy + 3y^2)\,dy + g(x)$
$= 2x^2y^4 + 3xy^2 + y^3 + g(x)$

then $\frac{\partial F}{\partial x} = 4xy^4 + 3y^2 + g'(x) = M(x,y) = 4xy^4 + 3y^2$
$\Rightarrow g'(x) = 0 \Rightarrow$ take $g(x) = 0$
and so $F(x,y) = 2x^2y^4 + 3xy^2 + y^3$

so the general solution is $2x^2y^4 + 3xy^2 + y^3 = C$

$y(1) = -1 \Rightarrow 2(1)^2(-1)^4 + 3(1)(-1)^2 + (-1)^3 = C \Rightarrow C = 4$

$\therefore$ the unique solution is $\boxed{2x^2y^4 + 3xy^2 + y^3 = 4}$

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6. $f(x) = x^3 - 4x + 2$ $\quad$ $\left.\begin{array}{l} f(0) = 2 \\ f(1) = -1 \end{array}\right\}$ $\therefore$ root in $[0,1]$

$f(x) = 0 \Rightarrow x^3 - 4x + 2 = 0 \Rightarrow x = \frac{2+x^3}{4}$

take $g(x) = \frac{2+x^3}{4}$, then $g'(x) = \frac{3}{4}x^2$

and so $|g'(x)| = \frac{3}{4}x^2 \le \frac{3}{4}$ on $[0,1]$

$\therefore$ this $g(x)$ will work

$x_0 = 0.5$, $\quad x_1 = g(x_0) = \frac{2+(0.5)^3}{4} = 0.53125$

$$\begin{aligned}
x_2 &= g(x_1) = 0.537483 \\
x_3 &= g(x_2) = 0.538818 \\
x_4 &= g(x_3) = 0.539108 \\
x_5 &= g(x_4) = 0.539171 \\
x_6 &= g(x_5) = 0.539185 \\
x_7 &= g(x_6) = 0.539188 \\
x_8 &= g(x_7) = 0.539189 = x_9 \quad \therefore \text{ stop}
\end{aligned}$$

(check: $f(0.539189) = (0.539189)^3 - 4(0.539189) + 2$
$= -3.98 \times 10^{-7}$ okay!)

$\therefore$ to 6 decimal places, the root is $\boxed{0.539189}$

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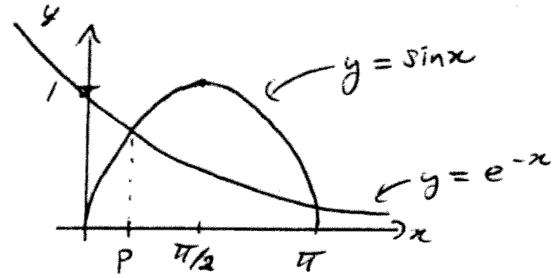
7. 

let $f(x) = e^{-x} - \sin x$, then $f'(x) = -e^{-x} - \cos x$

Newton's Method :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{e^{-x_n} - \sin(x_n)}{-e^{-x_n} - \cos(x_n)}$$

$$= x_n + \frac{e^{-x_n} - \sin(x_n)}{e^{-x_n} + \cos(x_n)}$$

$x_0 = \pi/4$, $x_1 = 0.569\,440$

$x_2 = 0.588\,389$

$x_3 = 0.588\,533 = x_4$ ∴ stop

(check : $e^{-0.588533} \approx 0.555141$

$\sin(0.588533) \approx 0.555141$ okay!)

∴ the solution is $\boxed{0.588533}$