UNIVERSITY OF WATERLOO
FINAL EXAMINATION
FALL 2014

Name (Please Print Legibly)	SOLUTIONS
Signature	
Student ID Number	

COURSE NUMBER	Math 117
COURSE TITLE	Calculus 1 for Engineering
COURSE SECTION(s)	001 002 003 004 005
DATE OF EXAM	December 4th, 2014
TIME PERIOD	12:30 - 3:00 pm
DURATION OF EXAM	2.5 hours
NUMBER OF EXAM PAGES	11
INSTRUCTORS (please check the appropriate box to tell us which section you are enrolled in)	
☐ **001 (ECE - M.Kohandel)**	☐ **002 (ECE - D.Harmsworth)**
☐ **003 (ECE - P.Sharma)**	☐ **004 (NE - D.Harmsworth)**
☐ **005 (SE - J.Penney)**	
EXAM TYPE	Closed Book
ADDITIONAL MATERIALS ALLOWED	None

Notes:

1. **Identify your instructor using the boxes above. This is worth 1 mark.**
2. Fill in your name, ID number and sign the paper.
3. Check that the examination has 11 pages.
4. Answer all questions in the space provided. Continue on the back of the *preceding* page if necessary.
5. **Your grade will be influenced by how clearly you express your ideas, and how well you organize your solutions.**

Marking Scheme:

Question	Mark	Out of
Identify Instructor		1
1		9
2		11
3		34
4		14
5		6
6		14
7		11
Total		100

[6] 1. a) For which values of x is the function $f(x) = x^2 H(x) + e^{1-x} H(x-1)$ differentiable?

$$f(x) = \begin{cases} 0 & , x < 0 \\ x^2 & , 0 \le x < 1 \\ x^2 + e^{1-x} & , x \ge 1 \end{cases}$$

Clearly differentiable for all $x \neq 0, 1$. (1)

For $x < 0$, $f'(x) = 0$

For $x > 0$, $f'(x) = 2x$,

So $\lim_{x \to 0} f'(x) = 0$.

Since f is also continuous at 0, this means it is differentiable at 0. (3)

At $x = 1$? $\lim_{x \to 1^-} f(x) = 1$, but $\lim_{x \to 1^+} f(x) = 2$,

so f is not cts. there, and hence is not diff'ble either. (2)

$\longrightarrow$ f is diff'ble for all $x \neq 1$.

[3] b) Suppose $f'(x) = \sqrt{x^3 + 1}$, and $f(0) = 5$. Write down an expression for $f(x)$ in terms of an integral.

Using the FTC, $f(x) = \int_0^x \sqrt{t^3+1}\, dt + C$.

Setting $f(0) = 5$ gives $C = 5$, so

$$f(x) = 5 + \int_0^x \sqrt{t^3+1}\, dt$$

(1) for $\int_a^x \sqrt{t^3+1}\, dt$

(1) for using $a = 0$.

(1) for introducing and evaluating C.

[11] 2. Evaluate the following limits, if possible:

a) $\lim_{x\to\infty} [\ln(10x+1) - \ln(6x+1)]$

$= \lim_{x\to\infty} \ln\left[\frac{10x+1}{6x+1}\right]$ (2)

now, $\frac{10x+1}{6x+1} \to \frac{5}{3}$ as $x\to\infty$, so (1)

$\ln\left[\frac{10x+1}{6x+1}\right] \to \underline{\ln\frac{5}{3}}$ as $x\to\infty$. (1)

b) $\lim_{x\to 0}\frac{\cosh x - 1}{\cos 2x - 1}$

$\overset{H}{=} \lim_{x\to 0}\frac{\sinh x}{-2\sin 2x}$ (2)

$\overset{H}{=} \lim_{x\to 0}\frac{\cosh x}{-4\cos 2x}$ (2)

$= \frac{-1}{4}.$ (1)

c) $\lim_{x\to 1}\frac{\sin^{-1}(x)}{x+2}$

$= \frac{\pi/2}{3} = \frac{\pi}{6}$ (2)

3. Evaluate the following integrals:

[7] a) $\displaystyle\int_0^2 \frac{2x+3}{x^2+3x+5}\,dx$

let $u = x^2+3x+5$
$du = (2x+3)\,dx$ 2

$= \displaystyle\int_5^{15} \frac{du}{u}$ 2

$= \ln u \Big|_5^{15}$ 1

$= \ln 15 - \ln 5$ 1

$= \ln 3.$ 1

[7] b) $\displaystyle\int_0^{\ln 2} xe^{2x}\,dx$

Use $u = x$, $dv = e^{2x}\,dx$ 2
$du = dx$, $v = \frac{1}{2}e^{2x}$ 1

$= uv - \int v\,du$

$= \frac{1}{2}xe^{2x}\Big|_0^{\ln 2} - \frac{1}{2}\displaystyle\int_0^{\ln 2} e^{2x}\,dx$ 2

$= \frac{1}{2}(\ln 2)e^{2\ln 2} - \frac{1}{4}e^{2x}\Big|_0^{\ln 2}$ 1

$= 2\ln 2 - \frac{1}{4}(4-1)$

$= 2\ln 2 - \frac{3}{4}.$ 1 (must be simplified)

[10] c) $\displaystyle\int \frac{4x^2+3}{x^2\left(x^2+1\right)}\,dx$

$$\frac{4x^2+3}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$ 3

$$4x^2+3 = Ax(x^2+1) + B(x^2+1) + x^2(Cx+D)$$ 1

$x=0$ gives $3=B$ 1

x^3 terms give $0=A+C$ 1

x^2 terms give $4=B+D$, so $D=1$ 1

x terms give $0=A$, so $C=0$. 1

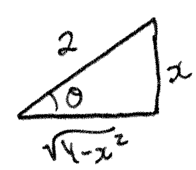
$$= \int\left(\frac{3}{x^2} + \frac{1}{x^2+1}\right)dx$$

$$= -\frac{3}{x} + \tan^{-1}x + C.$$ 2

[10] d) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

Let $x = 2\sin\theta$ 2

$dx = 2\cos\theta\, d\theta$ 1



$$= \int \frac{4\sin^2\theta\,(2\cos\theta\, d\theta)}{\sqrt{4-4\sin^2\theta}}$$

$$= 4\int \sin^2\theta\, d\theta$$ 2

$$= 2\int(1-\cos 2\theta)\,d\theta$$ 1

$$= 2\theta - \sin 2\theta + C$$ 1

$$= 2\theta - 2\sin\theta\cos\theta + C$$ 1

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C$$ 2

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}x\sqrt{4-x^2} + C$$

[14]

4. Sketch the graph of the function $f(x) = \frac{x+3}{\sqrt{x^2+1}}$, given that $f'(x) = \frac{1-3x}{(x^2+1)^{3/2}}$, $f''(x) = \frac{3(2x^2-x-1)}{(x^2+1)^{5/2}}$, and $\sqrt{2} \approx 1.4$.

Include any x- and y-intercepts, horizontal or vertical asymptotes, local extrema, or inflection points.

Note: to locate certain points on the graph you will need to approximate values of square roots, and you don't have a calculator. You know what to do; make a rough approximation using differentials!

$y=0 \Rightarrow x=-3.$
$x=0 \Rightarrow y=3$ (1)

No vertical asymptotes

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+3/x}{\sqrt{1+1/x}} = 1$ (1)

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1+3/x}{-\sqrt{1+1/x}} = -1$ (1)

$f'(x)=0 \Rightarrow x = 1/3$ (1)

$f''(x)=0 \Rightarrow 2x^2-x-1=0$

$\Rightarrow x = \frac{1\pm\sqrt{1+8}}{4} = 1, -1/2$ (2)

y values? At $x=\frac{1}{3}$, $f(x) = \frac{10/3}{\sqrt{10/9}} = \sqrt{10}$. (1)

Using $g(x)=\sqrt{x}$ and $x=9$, we get $\Delta g \approx g'(x)\Delta x$

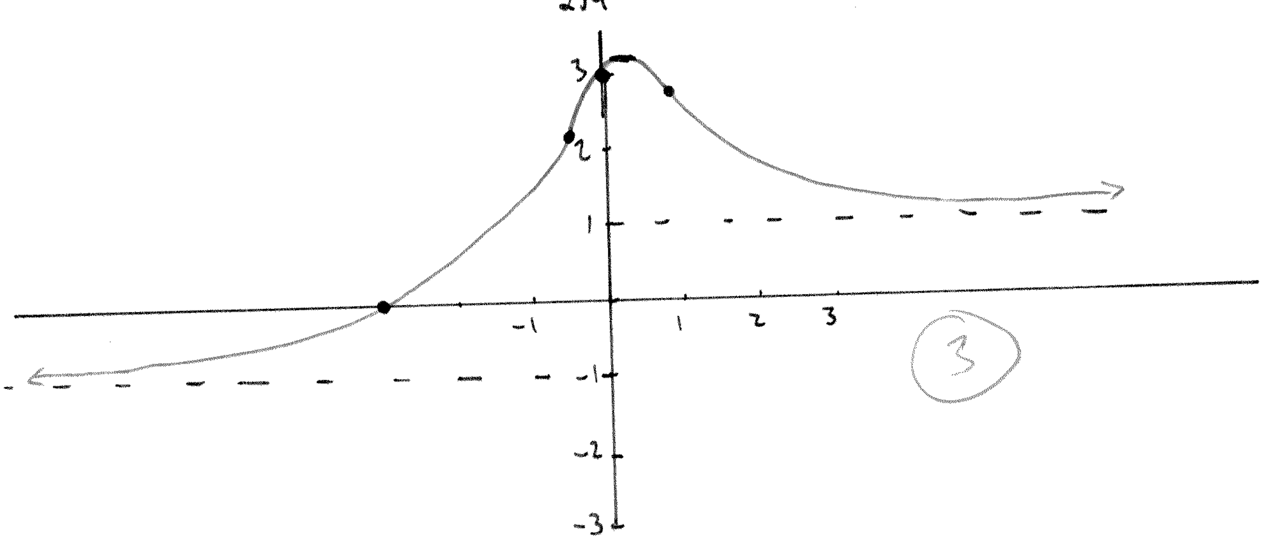
$= \frac{1}{2\sqrt{9}}(1) = \frac{1}{6}$,

so $\sqrt{10} \approx 3 + 1/6$ (2)

At $x=1$, $f(x) = \frac{4}{\sqrt{2}} = 2\sqrt{2} \approx 2.8$ (1)

At $x=-\frac{1}{2}$, $f(x) = \frac{5/2}{\sqrt{5/4}} = \sqrt{5}$

This time $\Delta g \approx \frac{1}{2\sqrt{4}}(1) = 0.25$, so $f(-\frac{1}{2}) \approx 2.25$. (1)



(3)

[6] 5. The inverse secant function is not defined the same way in every textbook. Some authors define it this way:

$$\boxed{y = \sec^{-1} x \text{ means that } x = \sec y \text{ and } y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right].}$$

Given *this* definition, find the formula for the derivative, $\frac{d}{dx}\left(\sec^{-1} x\right)$.

$x = \sec y \Rightarrow 1 = \sec y \tan y \frac{dy}{dx}$,

so $\frac{dy}{dx} = \frac{1}{\sec y \tan y}$ 2

Now, $\sec y = x$, and $\tan y = \pm\sqrt{\sec^2 y - 1}$, 2 for $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$

1 more for ($\pm$)

but $\tan y > 0$ on $\left[0, \frac{\pi}{2}\right)$ while $\tan y < 0$ on $\left(\frac{\pi}{2}, \pi\right]$,

Note that $\sec y > 0$ on $\left[0, \frac{\pi}{2}\right)$, while $\sec y < 0$ on $\left(\frac{\pi}{2}, \pi\right]$ also,

So $\frac{dy}{dx} = \begin{cases} \frac{1}{x\sqrt{x^2-1}} & \text{when } x > 0 \\ \frac{-1}{x\sqrt{x^2-1}} & \text{when } x < 0 \end{cases}$ 1

That is, $\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$.

6. Consider the curve with equation $x^4 + y^2 = 1$.

[4] a) Find any points at which the tangent line is horizontal or vertical.
Note: to locate points with vertical tangents, determine where $dx/dy = 0$.

$4x^3 + 2y\frac{dy}{dx} = 0$. $\frac{dy}{dx} = 0 \Rightarrow x = 0$, so $y = \pm 1$ $\rightarrow$ horiz. at $(0, \pm 1)$. 2

$4x^3\frac{dx}{dy} + 2y = 0$. $\frac{dx}{dy} = 0 \Rightarrow y = 0$, so $x = \pm 1$. $\rightarrow$ vert. at $(\pm 1, 0)$ 2

[10] b) By converting the equation into polar coordinates, locate the points on the curve which are furthest from the origin. That is, determine where $r(\theta)$ takes on its absolute maximum value on the inteval $[0, 2\pi)$. What is this maximum distance?

$r^4\cos^4\theta + r^2\sin^2\theta = 1$ 2

$4r^3r'\cos^4\theta - 4r^4\cos^3\theta\sin\theta + 2rr'\sin^2\theta + 2r^2\sin\theta\cos\theta = 0$

Set $r'(\theta) = 0$: $2r^2\sin\theta\cos\theta = 4r^4\cos^3\theta\sin\theta$ 2

$\Rightarrow \sin\theta = 0$ OR $\cos\theta = 0$ OR $2r^2\cos^2\theta = 1$

$\sin\theta = 0 \Downarrow \theta = 0, \pi \Downarrow r = 1$ 1

$\cos\theta = 0 \Downarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2} \Downarrow r = 1$ 1

$2r^2\cos^2\theta = 1 \Downarrow 2x^2 = 1$ 2 $\Downarrow x = \pm\frac{1}{\sqrt{2}}$, $y = \pm\frac{\sqrt{3}}{2}$, 1

and $r = \sqrt{x^2 + y^2} = \sqrt{\frac{1}{2} + \frac{3}{4}} = \frac{\sqrt{5}}{2}$ 1

[4] 7. a) When complex numbers are involved, the Squeeze Theorem becomes extremely useful. However, it does have to be modified, since inequalities are only meaningful for real numbers. What it states is this:

$$\text{If } |f(z)| \to 0, \text{ then } f(z) \to 0.$$

Show that if a and b are real numbers, with $a > 0$, then $\lim_{x\to\infty} e^{-(a+jb)x} = 0$.

Consider: $\left| e^{-(a+jb)x} \right| = \left| e^{-ax} e^{-jbx} \right|$ (1)

$= e^{-ax} \left| e^{-jbx} \right|$ (1)

$= e^{-ax} \left| \cos bx - j \sin bx \right|$ (1)

$= e^{-ax} \sqrt{\cos^2 bx + \sin^2 bx}$

$= e^{-ax}$ (1)

$\to 0$ as $x \to \infty$.

(7) b) In the study of heat transfer, it is possible to encounter integrals such as this:

$$I = \int_{-\infty}^{\infty} e^{-|x|} e^{j\omega x}\, dx.$$

In this expression x is a real variable, and ω is a real number. Find I.

$$I = \int_{-\infty}^{0} e^{x} e^{j\omega x} dx + \int_{0}^{\infty} e^{-x} e^{j\omega x} dx \quad (2)$$

$$= \lim_{t\to-\infty} \int_{t}^{0} e^{(1+j\omega)x} dx + \lim_{t\to\infty} \int_{0}^{t} e^{(-1+j\omega)x} dx \quad (2)$$

$$= \lim_{t\to-\infty} \left. \frac{e^{(1+j\omega)x}}{1+j\omega} \right|_{t}^{0} + \lim_{t\to\infty} \left. \frac{e^{(j\omega-1)x}}{j\omega-1} \right|_{0}^{t} \quad (1)$$

$$= \lim_{t\to-\infty} \left(\frac{1}{1+j\omega} - \frac{e^{(1+j\omega)t}}{1+j\omega} \right) + \lim_{t\to\infty} \left(\frac{e^{(j\omega-1)t}}{j\omega-1} - \frac{1}{j\omega-1} \right)$$

$$= \frac{1}{1+j\omega} + \frac{1}{1-j\omega} \quad \text{(using part a)} \quad (1)$$

$$= \frac{2}{1+\omega^2}. \quad (1)$$