Week 5 Practice Problems MATH 203

 # 1 Given that

$$f(x) = x^{10} \cdot h(x)$$

where $h(-1) = 3$ and $h'(-1) = 6$. Calculate $f'(-1)$

 # 2 Find the derivative of

a) $f(x) = (1+\sqrt{x}) \cdot x^{5/2}x^{-2}$ **b)** $f(x) = (1+x^3) \cdot e^{3x}$ **c)** $f(x) = \dfrac{x-\sqrt{x}}{x^2}$

d) $f(x) = (9x^2-5)^5(8x^2+6)^{14}$ **e)** $f(x) = \dfrac{x}{x+\frac{c}{x}}$

 # 3 Find the second derivative of

$$f(x) = xe^x(1+e^{-x})$$

 # 4 Find the equations to both lines through the point $(2, 2)$ that are tangent to the parabola

$$y = x^2 + x + 5$$

 # 5 Find the remaining trigonometric ratios for

a) $\sin(\theta) = \dfrac{3}{5}$, $0 < \theta < \dfrac{\pi}{2}$ **b)** $\sec(\phi) = -1.5$, $\dfrac{\pi}{2} < \phi < \pi$

Bonus Question

Give an example of a function $f(x)$ for which $f'(0)$ exists but $f''(0)$ does not, or explain why this is impossible.

Solutions:

 # 1

Recall the Product Rule:

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

For $f(x) = x^{10} \cdot h(x)$, where $h(-1) = 3$ and $h'(-1) = 6$. The derivative of f(x)

$$\begin{aligned} f'(x) &= 10x^9 \cdot h(x) + x^{10} \cdot h'(x) \\ f'(-1) &= 10(-1)^9 \cdot h(-1) + (-1)^{10} \cdot h'(-1) \\ &= -10 \cdot 3 + 1 \cdot 6 \\ &= -24 \end{aligned}$$

 # 2

a) (Midterm Winter 2015 #5 a))

$$\begin{aligned} f(x) &= (1 + \sqrt{x}) \cdot x^{5/2} \cdot x^{-2} \\ &= (1 + \sqrt{x}) \cdot x^{5/2-2} \\ &= (1 + \sqrt{x}) \cdot \sqrt{x} \\ &= \sqrt{x} + x \end{aligned}$$

$$\begin{aligned} f'(x) &= (x^{1/2})' + (x)' \\ &= \frac{1}{2} \cdot x^{1/2-1} + 1 \\ &= \frac{1}{2\sqrt{x}} + 1 \end{aligned}$$

b) (Midterm Winter 2014 #5 b))

$$\begin{aligned} f(x) &= (1 + x^3) \cdot e^{3x} \\ &= e^{3x} + x \cdot e^{3x} \end{aligned}$$

By applying 'Chain Rule' for e^{3x} we get $\frac{d}{dx}e^{3x} = 3 \cdot e^{3x}$. Now we apply the 'Product Rule' for the second term of $f(x)$.

$$\begin{aligned} f'(x) &= 3 \cdot e^{3x} + [3x^2 \cdot e^{3x} + x^3 \cdot 3 \cdot e^{3x}] \\ &= 3e^{3x}(1 + x^2 + x^3) \end{aligned}$$

c) (pg.188 #19, Stewart) You may apply Quotient Rule for this function but instead we will first divide, then compute the derivative term by term.

$$f(x) = \frac{x - \sqrt{x}}{x^2} = \frac{x}{x^2} - \frac{\sqrt{x}}{x^2} = \frac{1}{x} - \frac{1}{x^{3/2}} = x^{-1} - x^{-3/2}$$

$$f'(x) = -1 \cdot x^{-1-1} + \frac{3}{2} \cdot x^{-3/2-1} = -\frac{1}{x^2} + \frac{3}{2 \cdot x^{5/2}} = -x^{-2} + \frac{3}{2}x^{-5/2}$$

d) For this problem, we will use 'Chain Rule' then the 'Product Rule'. Recall the Chain Rule:

$$(f(g(x))' = f'(g(x)) \cdot g'(x)$$

So for,

$$f(x) = (9x^2 - 5)^5 \cdot (8x^2 + 6)^{14}$$

Let $P(x) = (9x^2 - 5)^5$ and $Q(x) = (8x^2 + 6)^{14}$. By Chain Rule,

$$P'(x) = 5 \cdot (9x^2 - 5)^4 \cdot 18x = 90x \cdot (9x^2 - 5)^4 \quad \text{and} \qquad Q'(x) = 14 \cdot (8x^2 + 6)^{13} \cdot 16x = 224x \cdot (8x^2 + 6)^{13}$$

Now, we use 'Product Rule' to compute $f'(x)$

$$\begin{aligned} f'(x) &= P'(x) \cdot Q(x) + P(x) \cdot Q'(x) \\ &= 224x \cdot (9x^2 - 5)^5 \cdot (8x^2 + 6)^{13} + 90x \cdot (8x^2 + 6)^{14} \cdot (9x^2 - 5)^4 \end{aligned}$$

You may simplify the expression but it is not necessary.

e) (pg.188 # 25, Stewart) First let us compute the derivative of the numerator and denominator separately,

$$\frac{d}{dx}\, x = 1 \qquad\qquad \frac{d}{dx}\left(x + \frac{c}{x}\right) = 1 - \frac{c}{x^2}$$

Recall Quotient Rule,

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

Apply this rule to compute $f'(x)$

$$f'(x) = \frac{\left(x + \frac{c}{x}\right) \cdot 1 - x \cdot \left(1 - \frac{c}{x^2}\right)}{\left(x + \frac{c}{x}\right)^2}$$

Again, you may simplify if you want.

 # 3 (Midterm Winter 2014 # 4)

$$\begin{aligned} f(x) &= x \cdot e^x (1 - e^{-x}) = x \cdot e^x - x \cdot e^x \cdot e^{-x} \\ &= x \cdot e^x - x \end{aligned}$$

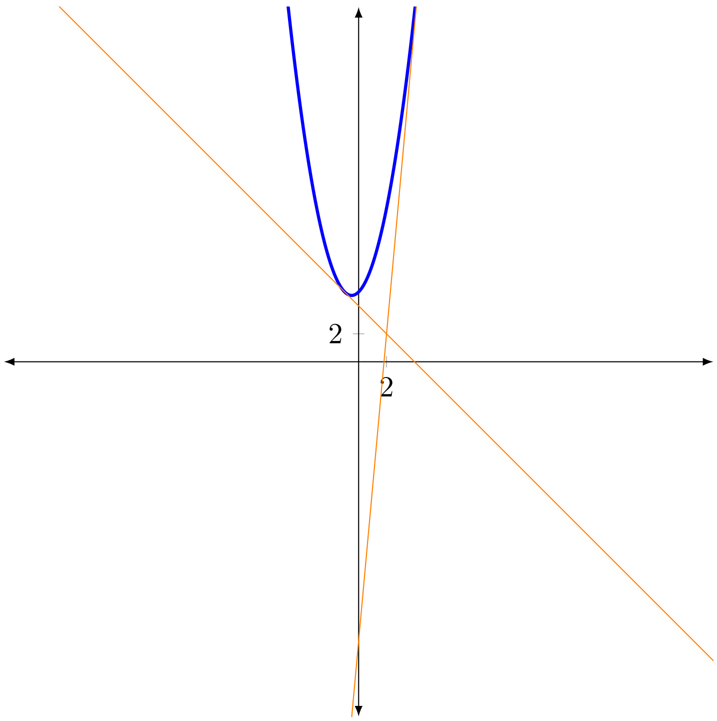
Apply Product Rule for $x \cdot e^x$,

$$f'(x) = e^x + x \cdot e^x - 1$$

$$f''(x) = x \cdot e^x + e^x + e^x = x \cdot e^x + 2 \cdot e^x$$

 # 4

We want to find two linear equations that are tangent to $f(x) = x^2 + x + 5$ and passes through the point $(2, 2)$. Graphically,



we want to find the equation of the orange lines that passes through $(2, 2)$.

Let

$$l_1 : y = m_1 x + b_1$$
$$l_2 : y = m_2 x + b_2$$

represent the equation of the tangent lines, where m_1 and m_2 are the slopes of the line l_1 and l_2, respectively. Let $(a, f(a))$ be the point of tangency (i.e, the points where the line and the curve $f(x)$ intersect).

By the definition, the slope of the tangent lines are equal to the derivative of $f(x)$ with $x = a$. In other words,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

Also, by the point slope formula

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

where the line passes through the points (x_0, y_0) and (x_1, y_1)

So, the derivative of $f(x)$ with $x = a$,

$$f'(a) = 2a + 1 = m$$

Since the lines passes through the point $(a, f(a))$ and $(2, 2)$, by the point slope formula

$$m_{1,2} = \frac{f(a) - 2}{a - 2} = 2a + 1$$

$$m_{1,2} = \frac{a^2 + a + 5 - 2}{a - 2} = 2a + 1$$

(Replace (x_0, y_0) and (x_1, y_1) with $(a, f(a))$ and $(2, 2)$). Now, solve for a

$$\begin{aligned}
\frac{a^2 + a + 3}{a - 2} &= 2a + 1 \\
a^2 + a + 3 &= (2a + 1)(a - 2) \\
a^2 + a + 3 &= 2a^2 - 3a - 2 \\
0 &= a^2 - 4a - 5 \\
0 &= (a - 5)(a + 1)
\end{aligned}$$

Hence, $a = -1$ and $a = 5$. So the slopes of each line,

$$\begin{aligned}
m_1 &= 2a + 1 = 2(-1) + 1 = -1 \\
m_2 &= 2a + 1 = 2(5) + 1 = 11
\end{aligned}$$

The equation of both tangent lines are

$$\begin{aligned}
l_1 &: y = -1x + b_1 \\
l_2 &: y = 11x + b_2
\end{aligned}$$

Now, we just have to find b_1 and b_2 (the y-intercept of the lines). Since both lines passes through $(2, 2)$, we have

$$\begin{aligned}
l_1 : y &= -x + b_1 \\
2 &= -2 + b_1 \\
b_1 &= 4
\end{aligned}$$

$$\begin{aligned}
l_2 : y &= 11x + b_2 \\
2 &= 11(2) + b_2 \\
b_2 &= -20
\end{aligned}$$

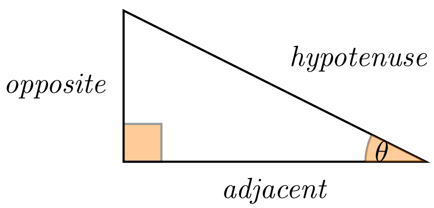
Therefore, the equation of both tangent lines are

$$l_1 : y = -x + 4$$
$$l_2 : y = 11x - 20$$

 # 5 (pg. A32 # 29; #31, Stewart)

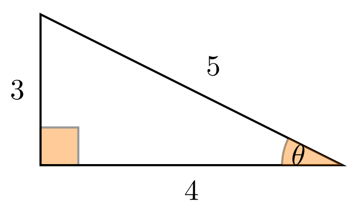
We want to find $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$ and $\cot\theta$ for each question.

a) Since $0 < \theta < \pi/2$, we can use SOHCAHTOA for right angled triangles.



$$\sin\theta = \frac{\text{opposite side}}{\text{hypothenuse side}} = \frac{3}{5}, \quad \text{where} \quad \text{opposite side} = 3 \quad \text{and} \quad \text{hypotenuse side} = 5$$

By 'Pythagorean Theorem', we have the adjacent side equal to 4.

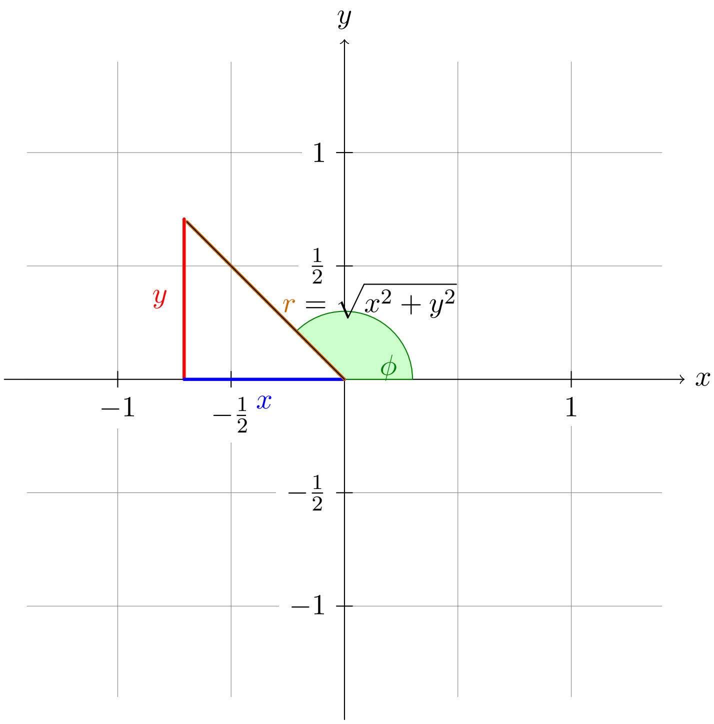


So,

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse side}} = \frac{4}{5}, \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3}{4}, \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{5}{3},$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{5}{4}, \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{4}{3},$$

b) Since $\pi/2 < \phi < \pi$, we can't use SOHCAHTOA for right angled triangle but now consider,



where,

$$\sin\phi = \frac{y}{r}, \qquad \cos\phi = \frac{x}{r} \qquad \text{and} \qquad r = \sqrt{x^2 + y^2}$$

Then

$$\sec\phi = -1.5 = -\frac{3}{2} = \frac{1}{\cos\phi} \Rightarrow \cos\phi = -\frac{2}{3},$$

Since r is always positive (because it's a length) and x lies on the 4th quadrant, then

$x = -2$ and $r = 3$. Now, to solve for y with

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ 3 &= \sqrt{(-2)^2 + y^2} \\ 9 &= y^2 + 4 \\ 5 &= y^2 \\ \sqrt{5} &= y \end{aligned}$$

y is positive because it's above the x-axis. Therefore,

$$\sin\phi = \frac{y}{r} = \frac{\sqrt{5}}{3}, \qquad \tan\phi = -\frac{\sqrt{5}}{2}, \qquad \csc\phi = \frac{3}{\sqrt{5}}, \qquad \cot\phi = -\frac{2}{\sqrt{5}}.$$

Bonus Question (Midterm Winter 2014 Bonus)

A nice example of a function that is undefined (does not exist) is when the denominator of a fraction equals to zero. Another undefined function would be a negative number inside a square root.

Since we have $x = 0$, take $f(x) = x^{3/2}$, then

$$f'(x) = \frac{3}{2}x^{1/2} \qquad\qquad f''(x) = \frac{3}{4}x^{-1/2} = \frac{3}{4 \cdot \sqrt{x}}$$

$f'(0) = 0$ exists but

$$f''(0) = \frac{3}{4 \cdot \sqrt{0}}$$

is undefined since the denominator equals to zero. Therefore $f''(0)$ does not exist.