

Week 4 Practice Problems MATH 203

1 The population of a colony of rabbits grows exponentially. The colony begins with 20 rabbits; 5 years later there are 400 rabbits.

- (a) Express the population of the colony of rabbits, P , as a function of time, t , in years.
- (b) Estimate how many years it will take for the population to reach 1000 rabbits.

2 What is the domain of $y = \ln(x^2 + x - 2)$?

3 Solve for x :

(a) $\frac{\log(x^3) + \log(x^5)}{\log(16x)} = 5$ (b) $2^{2x+17} = 3^{x-9}$

4 Find the range, the inverse function and the range of the inverse function for

a) $f(x) = \log_5(2x - 1)$ b) $g(x) = \frac{1 + e^x}{1 - e^x}$

c) $h(x) = e^{2x} + 2e^x$ d) $f(x) = (8 + 2^x)^{1/3}$

Solutions:

1

a) Since the population grows exponentially, it can be described by $P = ab^t$, where P is the number of rabbits and t is the number of years which have passed. We know that a represents the initial number of rabbits, so $a = 20$ and $P = 20(b)^t$. After 5 years, there are 400 rabbits so

$$\begin{aligned}400 &= 20(b)^5 \\20 &= b^5 \\(b^5)^{1/5} &= 20^{1/5} \\b &\approx 1.821\end{aligned}$$

From this, we know that $P(t) = 20(1.821)^t$.

b) We want to find t when $P = 1000$. Then

$$\begin{aligned}P(t) &= 1000 \\20(1.821)^t &= 1000 \\1.821^t &= 50\end{aligned}$$

Take the logarithm of both sides of the equal sides

$$\begin{aligned}\ln(1.821^t) &= \ln(50) \\t * \ln(1.821) &= \ln(50) \\t &= \frac{\ln(50)}{\ln 1.821} \\t &\approx 6.53\end{aligned}$$

2

We know anything inside a logarithm has to be strictly greater than zero, (i.e., $x^2+x-2 > 0$)

$$\begin{aligned}x^2 + x - 2 &> 0 \\x^2 + x &> 2\end{aligned}$$

By completing the square,

$$\begin{aligned}x^2 + x + \frac{1}{4} &> 2 + \frac{1}{4} = \frac{9}{4} \\ \left(x + \frac{1}{2}\right)^2 &> \frac{9}{4}\end{aligned}$$

Square root both sides to obtain two inequalities

$$\begin{aligned}x + \frac{1}{2} &> \frac{3}{2} & x + \frac{1}{2} &< -\frac{3}{2} \\ x &> 1 & x &< -2\end{aligned}$$

Hence, the domain of $\ln(x^2 + x - 2)$,

$$(-\infty, -2) \cup (1, \infty)$$

3

a) First we can multiply both sides by $\log(16x)$ and get:

$$\begin{aligned}\log(x^3) + \log(x^5) &= 5 \log(16x) \\ \log(x^{3+5}) &= 5 \log(16x) \\ \log(x^8) &= \log((16x)^5)\end{aligned}$$

Now we can exponentiate both sides and we get the following equation which we solve for x :

$$\begin{aligned}x^8 &= (16x)^5 \\ x^8 &= (16^5)x^5 \\ \frac{x^8}{x^5} &= (16^5) \\ x^{8-5} &= (16)^5 \\ x^3 &= (16)^5 \\ x &= (16)^{5/3}\end{aligned}$$

b)

$$2^{2x+17} = 3^{x-9}$$

Take the logarithm on both sides,

$$\ln(2^{2x+17}) = \ln(3^{x-9})$$

$$(2x + 17) \cdot \ln(2) = (x - 9) \cdot \ln(3)$$

$$(2x + 17) = (x - 9) \frac{\ln(3)}{\ln(2)}$$

$$2x + 17 = \frac{\ln 3}{\ln 2} \cdot x - 9 \cdot \frac{\ln 3}{\ln 2}$$

$$2x - \frac{\ln 3}{\ln 2} \cdot x = -17 - 9 \cdot \frac{\ln 3}{\ln 2}$$

$$\left(2 - \frac{\ln 3}{\ln 2}\right) \cdot x = \left(-17 - 9 \cdot \frac{\ln 3}{\ln 2}\right)$$

$$x = \frac{\left(-17 - 9 \cdot \frac{\ln 3}{\ln 2}\right)}{2 - \frac{\ln 3}{\ln 2}}$$

$$x \approx -75.3297$$

4

a) (Midterm Winter 2014 # 1c)

The range of $f(x)$ is all real numbers. To compute the inverse of $f(x)$, change y for x and x for y for $f(x)$, then solve for y

$$x = \log_5(2y - 1)$$

Exponentiate both sides by 5

$$\begin{aligned}5^x &= 5^{\log_5(2y-1)} \\5^x &= 2y - 1 \\ \frac{5^x + 1}{2} &= y\end{aligned}$$

Hence, $f^{-1}(x) = \frac{5^x + 1}{2}$. The range of f^{-1} is the domain of f , which is

$$\begin{aligned}2x - 1 &> 0 \\ x &> 1/2\end{aligned}$$

Hence the range of f^{-1} is

$$(1/2, \infty)$$

b)

First, let's find the inverse g^{-1} ,

$$\begin{aligned}x &= \frac{1 + e^y}{1 - e^y} \\ x \cdot (1 - e^y) &= (1 + e^y) \\ x - x \cdot e^y &= 1 + e^y \\ x - 1 &= x \cdot e^y + e^y \\ x - 1 &= e^y \cdot (x + 1) \\ \frac{x - 1}{x + 1} &= e^y\end{aligned}$$

Take the natural logarithm on both sides,

$$\ln\left(\frac{x - 1}{x + 1}\right) = y$$

Hence,

$$g^{-1}(x) = \ln\left(\frac{x - 1}{x + 1}\right)$$

Now, the range of g^{-1} is the domain of g , so we need

$$\frac{x - 1}{x + 1} > 0$$

For a fraction to be always positive, then there are two cases. Either the numerator and denominator are both positive **OR** they are both negative. For the first case,

$$\begin{array}{ll} x - 1 > 0 & x + 1 > 0 \\ x > 1 & x > -1 \end{array}$$

We take $x > 1$ since it satisfies both inequalities. Similarly, for the second case

$$\begin{array}{ll} x - 1 < 0 & x + 1 < 0 \\ x < 1 & x < -1 \end{array}$$

This time we take $x < -1$. Therefore,

$$\text{Domain}_{g^{-1}} = (-\infty, -1) \cup (1, \infty) = \text{Range}_g$$

The domain of g is the range of g^{-1} . We can see that $g(x)$ is undefined when $x = 0$, since $e^0 = 1$ and in the denominator of g , we have $1 - e^0 = 0$. Thus,

$$\text{Domain}_g = (-\infty, 0) \cup (0, \infty) = \text{Range}_{g^{-1}}$$

c) **(Final Exam April 2015 # 1b)**

Replace $u = e^x$ then,

$$y = e^{2x} + 2e^x = u^2 + 2u$$

We can solve for u by 'completing the square' or use the 'Quadratic equation'. We will solve the problem by just completing the square.

$$\begin{aligned} u^2 + 2u &= y \\ u^2 + 2u + 1 &= y + 1 \\ (u + 1)^2 &= y + 1 \\ u + 1 &= \pm\sqrt{y + 1} \\ u &= -1 \pm \sqrt{y + 1} \end{aligned}$$

change u back to e^x , then take the natural logarithm on both sides,

$$e^x = -1 \pm \sqrt{y+1}$$
$$x = \ln(-1 \pm \sqrt{y+1})$$

Anything inside a logarithm has to be always positive, then

$$x = \ln(-1 + \sqrt{y+1}) = h^{-1}(x)$$

The range of h is $(0, \infty)$ and the range of h^{-1} are all real numbers.

d) **(Final Exam December 2014 1c)**

Inverse of f ,

$$x = (8 + 2^y)^{1/3}$$
$$x^3 = 8 + 2^y$$
$$x^3 - 8 = 2^y$$
$$\log_2(x^3 - 8) = y$$
$$f^{-1}(x) = \log_2(x^3 - 8)$$

The range of f is the domain of f^{-1} , which is

$$x^3 - 8 > 0$$
$$x^3 > 8$$
$$x > 2$$

Since the domain of f are all real numbers, then range of f^{-1} are all real numbers.