ENGR-213 FINAL EXAM (SAMPLE)

SOLUTIONS

This is just a sample; it gives an idea of the structure and the level of the exam.

1. Solve these equations using an appropriate substitution.

(a) $$\frac{dy}{dx}=1/(e^{x+2y}+e^{-x-2y})-1/2 \ ;$$

Solution: Substitution $x+2y=z;\quad dx+2dy=dz;\quad dy=\frac{dz-dx}{2};$

$$dz-dx=(2/(e^{z}+e^{-z})-1)dx;\qquad dz=\frac{2}{e^{z}+e^{-z}}dx;\qquad \int(e^{z}+e^{-z})dz=\int 2dx+C;$$

$e^{z}-e^{-z}=2x+C;$ Answer: $e^{x+2y}-e^{-x-2y}=2x+C.$

(b) $$4x^2\frac{dy}{dx}=4y^2+x^2 \ .$$

Solution: $\frac{dy}{dx}=\frac{4y^2+x^2}{4x^2}=\frac{y^2}{x^2}+\frac{1}{4};$ Homogeneous equation.

Substitution: $y=ux;\ dy=xdu+udx;\ xdu+udx=(u^2+\frac{1}{4})dx;$

$$xdu=(u^2-u+\frac{1}{4})dx;\qquad \frac{du}{(u-1/2)^2}=\frac{dx}{x};\quad \frac{\int du}{(u-1/2)^2}=\frac{\int dx}{x};$$

$$-\frac{1}{u-1/2}=\ln x+C;\qquad u=\frac{1}{2}-\frac{1}{\ln x+C};$$

Answer: $\frac{y}{x}=\frac{1}{2}-\frac{1}{\ln x+C}.$

2. Reduce to an exact equation and solve it:

(a) $$\frac{dy}{dx}=\frac{2x\sin 2y+(2\sin y)/x^2}{2x^2\cos x-(2\cos y)/x};$$

Sorry, wrong equation.

(b) $$(2x^2 y\sin y-xy^3\sin x)dx+(x^3\sin y+x^3 y\cos y+3xy^2\cos x)dy=0 .$$

Solution: $M=2x^2 y\sin y-xy^3\sin x;\quad N=x^3\sin y+x^3 y\cos y+3xy^2\cos x;$

$$\frac{\partial M}{\partial y}=2x^2\sin y+2x^2 y\cos y-3xy^2\sin x;\quad \frac{\partial N}{\partial x}=3x^2\sin y+3x^2 y\cos y+3y^2\cos x-3xy^2\sin x;$$

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=-x^2\sin y-x^2 y\cos y-3y^2\cos x ;$$
$$\frac{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)}{N}=-\frac{1}{x};$$

Hence, the integrating factor $\mu=e^{\int\frac{-dx}{x}}=\frac{1}{x};$ after multiplication by μ we have the exact equation

$$(2xy\sin y-y^3\sin x)dx+(x^2\sin y+x^2 y\cos y+3y^2\cos x)dy=0;$$

$$f(x,y)=\int(2xy\sin y-y^3\sin x)dx+g(y)=x^2 y\sin y+y^3\cos x+g(y);$$

$$\frac{\partial f}{\partial y}=x^2\sin y+x^2 y\cos y+3y^2\cos x+g'(y)=x^2\sin y+x^2 y\cos y+3y^2\cos x;$$

$$g'(y)=0;\quad g(y)=const;\quad f(x,y)=x^2 y\sin y+y^3\cos x;$$

Answer: $x^2 y\sin y+y^3\cos y=C.$

3. Solve the equation using the integrating factor:

$$(x+1)\frac{dy}{dx}+xy=e^{-x} \ .$$

Solution: $\frac{dy}{dx}+\frac{x}{x+1}y=\frac{e^{-x}}{x+1};$

Integrating factor $m=e^{\int \frac{x}{x+1}dx}=e^{\int (1-1/(x+1))dx}=e^{x}\cdot e^{-\ln(x+1)}=\frac{e^{x}}{x+1};$

$$\frac{d}{dx}\left(\frac{e^{x}}{x+1}y\right)=\frac{1}{(x+1)^2}; \quad \frac{e^{x}}{x+1}y=-\frac{1}{x+1}+C;$$

Answer: $y=-e^{-x}+Ce^{-x}(x+1).$

4. Solve the Bernoulli equation

$$\frac{dy}{dx}-\frac{1}{x}y+3y^2=0 \ .$$

Solution: Substitution $y=\frac{1}{z};$ $\frac{dy}{dx}=-\frac{1}{z^2}\frac{dz}{dx};$

$$\frac{-1}{z^2}\frac{dz}{dx}-\frac{1}{xz}+\frac{3}{z^2}=0; \qquad \frac{dz}{dx}+\frac{1}{x}z=3;$$

$$x\frac{dz}{dx}+z=3\text{x}; \qquad \frac{d}{dx}(xz)=3x; \qquad xz=\frac{3}{2}x^2+C;$$

$z=\frac{3}{2}x+\frac{C}{x};$ Answer: $y=\frac{1}{z}=\frac{1}{\frac{3}{2}x+\frac{C}{x}}.$

5.

(a) Find the general solution of the equation

$$x^2y''-3xy'+2y=x^2+\frac{1}{x} \;;$$

Solution: This is a Cauchy-Euler equation. The characteristic equation:

$$m(m-1)-3m+2=0; \quad m^2-4m+2=0; \qquad m_1=2+\sqrt{2},\; m_2=2-\sqrt{2}.$$

$$y_c=c_1x^{2+\sqrt{2}}+c_2x^{2-\sqrt{2}};$$

$y_p=ax^2+\dfrac{b}{x}$ (no resonances);

$$y_p'=2ax-\frac{b}{x^2}; \quad y_p''=2a+\frac{2b}{x^3}; \quad x^2y''-3xy'+2y=2ax^2+\frac{2b}{x}-6ax^2+\frac{3b}{x}+2ax^2+\frac{2b}{x}$$

$$=-2ax^2+\frac{7b}{x}; \qquad a=-\frac{1}{2}, \quad b=\frac{1}{7.}$$

General solution: $y=-\dfrac{1}{2}x^2+\dfrac{1}{7x}+c_1x^{2+\sqrt{2}}+c_2x^{2-\sqrt{2}}.$

(b) Solve the initial value problem

$$y''-3y'+2y=3e^{2t} \;; \qquad y(0)=2 \;; \quad y'(0)=1 \;.$$

Solution: Characteristic equation: $\lambda^2-3\lambda+2=0;\; \lambda_1=1,\; \lambda_2=2.$

$y_c=c_1e^t+c_2e^{2t};$ the right hand side is resonant, so the particular solution $y_p=ate^{2t}$. Substituting into the equation, we find $a=3$, and the general solution is

$y=3te^{2t}+c_1e^t+c_2e^{2t}$. Then the initial conditions give us $y(0)=c_1+c_2=2,$ $y'(0)=3+c_1+2c_2=1$; from here we have $c_1=-3, \quad c_2=4.$

Answer: $y=3te^{2t}-3e^t+4e^{2t}.$

6. Find the general solution of the equation

$$y''+2y'+2y=e^{-t}\sin t+e^{-2t}\cos 2t$$

Solution. Characteristic equation: $\lambda^2+2\lambda+2=0;$ $\lambda=-1\pm i.$ So, the general solution of the homogeneous equation is $y_c=e^{-t}(c_1\cos t+c_2\sin t).$ The first term in the right hand side is resonant, and the second one is not. So, the particular solution of the nonhomogeneous equation has the form $y_p=te^{-t}(d_1\cos t+d_2\sin t)+e^{-2t}(d_3\cos 2t+d_4\sin 2t)$. Substituting this into the equation and comparing with the right hand side, we find that $d_1=\frac{-1}{2},\ d_2=\frac{1}{2},\ d_3=\frac{-1}{5},\ d_4=\frac{-1}{10}$, and the answer is

$$y=te^{-t}\left(\frac{-1}{2}\cos t+\frac{1}{2}\sin t\right)+e^{-2t}\left(\frac{-1}{10}\cos 2t-\frac{1}{5}\sin 2t\right)+c_1e^{-t}\cos t+c_2e^{-2t}\sin 2t\ .$$

7. A mass $m=10kg$ is suspended on a spring and stretches it by $h=2cm$. Find the period of oscillation (the friction and air resistance are negligible; the gravity constant $g=9.81m/sec^2$).

Solution. The spring constant $k=\frac{F}{h}=\frac{mg}{h}=4905\ (N/m)$;

The frequency $\omega=\sqrt{\frac{k}{m}}$; the period $T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m}{k}}=2\pi\sqrt{\frac{10}{4905}}=0.2837\ (sec)$.

Answer: $T=0.2837\ sec$.

8. Solve the equation using the variation of parameters:

$$y''-2y'-8y=x$$

Solution. Characteristic equation: $\lambda^2-2\lambda-8=0;$ $\lambda=\frac{2\pm\sqrt{4+32}}{2};$ $\lambda_1=-2,\ \lambda_2=4$.

Solutions of the homogeneous equation: $y_1=e^{-2x},$ $y_2=e^{4x}$. The Wronskian

$$W=\begin{vmatrix} e^{-2x} & e^{4x} \\ -2e^{-2x} & 4e^{4x} \end{vmatrix}=6e^{2x} \ ;$$

$$y=u_1y_1+u_2y_2 \ ;$$

$$u_1'=\frac{-xe^{4x}}{ce^{2x}}=-\frac{1}{6}xe^{2x}; \qquad u_2'=\frac{xe^{-2x}}{6e^{2x}}=\frac{1}{6}xe^{-4x} \ ;$$

$$u_1=\int -\frac{1}{6}xe^{2x}dx=-\frac{1}{6}(\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x})+c_1 \ ;$$

$$u_2=\frac{1}{6}\int xe^{-4x}dx=\frac{1}{6}(-\frac{1}{4}xe^{-4x}-\frac{1}{16}e^{-4x})+c_2 \ ;$$

$$y=e^{-2x}\left(-\frac{1}{12}xe^{2x}+\frac{1}{24}e^{2x}+c_1\right)+e^{4x}\left(-\frac{1}{24}xe^{-4x}-\frac{1}{96}e^{-4x}+c_2\right)=-\frac{1}{8}x+\frac{1}{32}+c_1e^{-2x}+c_2e^{4x}.$$

9.

(a) Solve the initial value problem for the system

$$\frac{dx_1}{dt}=x_1-2x_2$$

$$\frac{dx_1}{dt}=x_1+3x_2$$

Initial condition: $x_1(0)=0$, $x_2(0)=-1$.

Solution. Solve this system by elimination. From the second equation,

$x_1=\frac{dx_2}{dt}-3x_2$. Substitute into the first equation:

$$\frac{d^2x_2}{dt^2}-3\frac{dx_2}{dt}-\frac{dx_2}{dt}+3x_2+2x_2=0\ ;$$

$$x_2''-4x_2'+5x_2=0\ ; \qquad \lambda^2-4\lambda+5=0; \qquad \lambda=2\pm i\ .$$

$x_2=e^{2t}(c_1\cos t+c_2\sin t)$. Substitute into the equation for x_1 :

$$x_1=\frac{dx_2}{dt}-3x_2=e^{2t}\left[(-c_1+c_2)\cos t+(-c_1-c_2)\sin t\right]\ .$$

Substitute into the initial conditions:

$$x_1(0)=c_1+c_2=0; \qquad c_1=c_2\ ;$$

$$x_2(0)=c_1=1; \qquad c_1=1,\ c_2=1\ .$$

Answer: $x_1=e^{2t}(-2\sin t);$ $\qquad x_2=e^{2t}(\cos t+\sin t)$.

(b) Find a general solution of the system

$$\frac{dx}{dt}=y+t$$

$$\frac{dy}{dt}=4\mathrm{x}-t$$

Solution. We solve this problem by diagonalization.

$A=\begin{pmatrix}0 & 1\\ 4 & 0\end{pmatrix}$; the system has the form $\frac{dX}{dt}=AX+F$ where $X=\begin{pmatrix}x\\ y\end{pmatrix},\quad F(t)=\begin{pmatrix}t\\ -t\end{pmatrix}.$

$$det(A-\lambda I)=det\begin{pmatrix}-\lambda & 1\\ 4 & -\lambda\end{pmatrix}=\lambda^2-4=0; \qquad \lambda_1=2, \quad \lambda_2=-2.$$

$$A-\lambda_1 I=\begin{pmatrix}-2 & 1\\ 4 & -2\end{pmatrix};$$ the first eigenvector $K_1=\begin{pmatrix}1\\2\end{pmatrix};$

$$A-\lambda_2 I=\begin{pmatrix}2 & 1\\ 4 & 2\end{pmatrix};$$ the second eigenvector $K_2=\begin{pmatrix}1\\2\end{pmatrix}$.

$$U=(K_1\ K_2)=\begin{pmatrix}1 & 1\\ 2 & -2\end{pmatrix}; \qquad U^{-1}=-\frac{1}{4}\begin{pmatrix}-2 & -1\\ -2 & 1\end{pmatrix}=\begin{pmatrix}1/2 & 1/4\\ 1/2 & -1/4\end{pmatrix};$$

The diagonal matrix of eigenvalues $D=\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}$; so, $A=UDU^{-1}$.

Our system has the form $\frac{dX}{dt}-AX=F;$ multiplying it by U^{-1} , we have

$\frac{dC}{dt}-DC=U^{-1}F$ where $C=U^{-1}X=\begin{pmatrix}c_1\\ c_2\end{pmatrix}$. Now,
$U^{-1}F=\begin{pmatrix}1/2 & 1/4\\ 1/2 & -1/4\end{pmatrix}\begin{pmatrix}t\\ -t\end{pmatrix}=\begin{pmatrix}t/4\\ 3t/4\end{pmatrix}$, and the system for c_1, c_2 is

$$\frac{dc_1}{dt}-2c_1=t/4;$$
$$\frac{dc_2}{dt}+2c_2=3t/4.$$

These two equations are independent from one another. Solving them by the undefined coefficients, we find

$c_1=-t/8-1/16+d_1e^{2t};$ $\quad c_2=3t/8-3/16+d_2e^{-2t},$ and from the equality $X=UC$
we find

$$x=c_1(t)+c_2(t)=t/4-1/4+d_1e^{2t}+d_2e^{-2t};$$

$$y=2c_1-2c_2=-t+1/4+2d_1e^{2t}-2d_2e^{-2t}.$$

10. Find the power series solution of the equation $(x^2+1)y''+y=0$; find the convergence radius of the series.

Solution. Let us look for the solution in the form $y=\sum_{n=0}^{\infty} c_n x^{n.}$. Then

$y''=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$. Substituting into the equation gives:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n+\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}+\sum_{n=0}^{\infty} c_n x^n=0.$$

Making index shift in the second term ($k=n-2;\quad n=k+2$), and renaming n by k in the first and the third terms, we get the equation

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k+\sum_{k=0}^{\infty} (k+1)(k+2)c_{k+2}x^k+\sum_{k=0}^{\infty} c_k x^k=0,$$

or

$$c_0+c_1x+2c_2+2\cdot 3c_3x+\sum_{k=2}^{\infty}\left[\ (k(k-1)+1)c_k\ +\ (k+1)(k+2)c_{k+2}\ \right]x^k=0\ .$$

The coefficients c_0 and c_1 are arbitrary (they are the initial conditions: $c_0=y(0),\quad c_1=y'(0)$). The further coefficients are found by equating the coefficients at x^k to zero for all k :

$k=0$: $\quad 2c_2+c_0=0.$

$k=1$: $\quad 2\cdot 3c_3+(1\cdot 2+1)c_2=0$

$k=2$: $\quad 3\cdot 4c_4+(1\cdot 2+1)c_2=0$

$k=3$: $\quad 4\cdot 5c_5+(2\cdot 3+1)c_3=0$

....

k: $\qquad (k+1)(k+2)c_{k+2}+(k\cdot(k-1))c_k=0$

From here we have the following chain of equalities:

$$c_2=-\frac{c_0}{2};\quad c_4=-\frac{1\cdot 2+1}{3\cdot 4}c_2;\qquad c_6=-\frac{3\cdot 4+1}{5\cdot 6}c_4;.........;\ c_{2k+2}=-\frac{(2k-1)\cdot 2k+1}{(2k+1)(2k+2)}c_{2k};.....$$

$$c_3 = -\frac{1}{2\cdot 3}c_1; \quad c_5 = -\frac{2\cdot 3+1}{4\cdot 5}c_3; \quad c_7 = -\frac{4\cdot 5+1}{6\cdot 7}c_5; \quad .... \quad ; c_{2n+1} = -\frac{(2n-2)(2n-1)+1}{2n(2n+1)}c_{2n-1}; \; ....$$

So, the first terms of the series are

$$y = c_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{13}{240}x^6 + ...\right) + c_1\left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5 - \frac{147}{5040}x^7 + \; ...\right).$$

To find the convergence radius, we use the Ratio Test.

$$\frac{c_{k+2}}{c_k} = -\frac{k(k-1)+1}{(k+1)(k+2)} = -\frac{1\cdot(1-1/k)+1/k^2}{(1+1/k)(1+2/k)} \to 1 \quad \text{as} \quad k \to \infty \; .$$

Hence, the convergence radius $R = 1$.