MAT2384 Assignment #6

Due Tuesday November 25 at the beginning of class.

Late assignments will not be accepted, nor will unstapled assignments.

Student Name ______________________________ Student Number ______________________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

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1. Find the Laplace transform of the following functions

   (a) $f(t) = \cosh(3t) - 2e^{-3t} + 1$

$$\begin{aligned}\mathcal{L}[\cosh(3t) - 2e^{-3t} + 1] &= \mathcal{L}[\cosh(3t)] - 2\mathcal{L}[e^{-3t}] + \mathcal{L}[1] \\ &= \frac{s}{s^2 - 9} - \frac{2}{s+3} + \frac{1}{s}\end{aligned}$$

   (b) $g(t) = 3t^3 - 5t^2 + t + 5$

$$\begin{aligned}\mathcal{L}[3t^3 - 5t^2 + t + 5] &= 3\mathcal{L}[t^3] - 5\mathcal{L}[t^2] + \mathcal{L}[t] + \mathcal{L}[5] \\ &= 3\frac{3!}{s^4} - 5\frac{2!}{s^3} + \frac{1}{s^2} + 5\frac{1}{s}\end{aligned}$$

   (c) $h(t) = 2\sin(-3t) + 3\cos(-3t)$

$$\begin{aligned}\mathcal{L}[2\sin(-3t) + 3\cos(-3t)] &= -2\mathcal{L}[\sin(3t)] + 3\mathcal{L}[\cos(3t)] \\ &= -2\frac{3}{s^2+9} + 3\frac{s}{s^2+9} \\ &= \frac{-6+3s}{s^2+9}\end{aligned}$$

2. Find the inverse Laplace transform of the following functions.

   (a) $F(s) = \dfrac{2s+3}{s^2 - 4s + 3}$

   We have [1]

$$\begin{aligned}\frac{2s+3}{s^2-4s+3} &= \frac{2s+3}{(s-3)(s-1)} \\ \frac{2s+3}{(s-3)(s-1)} &= \frac{A}{s-3} + \frac{B}{s-1} \\ 2s+3 &= A(s-1) + B(s-3)\end{aligned}$$

$s = 1$ $\qquad 5 = B(-2) \qquad B = -\frac{5}{2}$

$s = 3$ $\qquad 9 = A(2) \qquad A = \frac{9}{2}$

$$\begin{aligned}\frac{2s+3}{(s-3)(s-1)} &= \frac{9}{2(s-3)} - \frac{5}{2(s-1)} \\ \mathcal{L}^{-1}\left[\frac{9}{2(s-3)} - \frac{5}{2(s-1)}\right] &= \frac{9}{2}e^{3t} - \frac{5}{2}e^{t}\end{aligned}$$

(b) $G(s) = \dfrac{s+1}{s^2+25}$

$$\frac{s+1}{s^2+25} = \frac{s}{s^2+25} + \frac{1}{s^2+25}$$
$$\mathcal{L}^{-1}\left[\frac{s+1}{s^2+25}\right] = \cos 5t + \frac{1}{5}\sin 5t$$

3. Use the Laplace transform to solve the following initial value problems.

(a) $y'' + 5y' - 6y = 0$, $y(0) = -1$, $y'(0) = 3$

Taking the Laplace transform, we have [2]

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 5s\mathcal{L}[y] - 5y(0) - 6\mathcal{L}[y] = 0$$
$$s^2\mathcal{L}[y] + s - 3 + 5s\mathcal{L}[y] - 5y(0) - 6\mathcal{L}[y] = 0$$
$$(s^2 + 5s + 6)\mathcal{L}[y] = -s - 2$$
$$\mathcal{L}[y] = \frac{-s-2}{(s+6)(s-1)}$$

Using partial fractions, we have

$$\frac{-s-2}{(s+6)(s-1)} = \frac{A}{s+6} + \frac{B}{s-1}$$
$$-s - 2 = A(s-1) + B(s+6)$$

$s = 1$ $\qquad -3 = B(7)$ $\qquad B = -\dfrac{3}{7}$

$s = -6$ $\qquad 4 = A(-1)$ $\qquad A = -\dfrac{4}{7}$

Hence

$$\mathcal{L}[y] = -\frac{4}{7}\frac{1}{s+6} - \frac{3}{7}\frac{1}{s-1}$$
$$y = -\frac{4}{7}e^{-6t} - \frac{3}{7}e^{t}$$

(b) $y'' + 4y' + 3y = e^{-t}$, $y(0) = -1$, $y'(0) = 2$

We have [2]

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 4s\mathcal{L}[y] - 4y(0) + 3\mathcal{L}[y] = \mathcal{L}[e^{-t}]$$
$$s^2\mathcal{L}[y] + s - 2 + 4s\mathcal{L}[y] + 4 + 3\mathcal{L}[y] = \frac{1}{s+1}$$
$$(s^2 + 4s + 3)\mathcal{L}[y] = \frac{1}{s+1} - s - 2$$
$$(s+1)(s+3)\mathcal{L}[y] = \frac{-s^2 - 3s - 1}{s+1}$$
$$\mathcal{L}[y] = \frac{-s^2 - 3s - 1}{(s+1)^2(s+3)}$$

Using partial fractions, we have

$$\frac{-s^2 - 3s - 1}{(s+1)^2(s+3)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3}$$
$$-s^2 - 3s - 1 = A(s+1)(s+3) + B(s+3) + C(s+1)^2$$

$s = -1$ $\qquad 1 = 2B$ $\qquad B = \dfrac{1}{2}$

$s = -3$ $\qquad -1 = 4C$ $\qquad C = -\dfrac{1}{4}$

$s = 0$ $\qquad -1 = 3A + \dfrac{3}{2} - \dfrac{1}{4}$ $\qquad A = -\dfrac{3}{4}$

We thus have

$$\mathcal{L}[y] = -\frac{3}{4}\frac{1}{s+1} + \frac{1}{2}\frac{1}{(s+1)^2} - \frac{1}{4}\frac{1}{s+3}$$
$$y = -\frac{3}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}$$

using t-shifting for the middle term.

4. (a) Use Gaussian quadrature of order 4 to estimate the value of the integral

$$\int_0^3 \frac{1}{1+x}dx$$

Compute the exact value of the integral and find the error. [3]

First we need to transform the integral limits. Let $u = a(x+b)$. Then

$$\begin{aligned} u(0) &= ab = -1 \\ u(3) &= a(3+b) = 1 \\ a &= \frac{1}{3+b} \\ \frac{b}{3+b} &= -1 \\ b &= -3 - b \\ 2b &= -2 \\ b &= -\frac{3}{2} \\ a &= \frac{2}{3} \end{aligned}$$

The substitution is then

$$\begin{aligned} u &= \frac{2}{3}\left(x - \frac{3}{2}\right) \\ \frac{du}{dx} &= \frac{2}{3} \\ dx &= \frac{3}{2}du \\ \frac{3}{2}u &= x - \frac{3}{2} \\ x &= \frac{3}{2}u + \frac{3}{2} \end{aligned}$$

We thus have

$$\begin{aligned} \int_{-1}^{1} \frac{1}{1+\frac{3}{2}u+\frac{3}{2}} \cdot \frac{3}{2}du &= \frac{3}{2}\int_{-1}^{1} \frac{1}{\frac{5}{2}+\frac{3}{2}u}du \\ &= \frac{3}{2}\left[0.6521451549\left(\frac{1}{\frac{5}{2}+\frac{3}{2}(0.3399810436)}\right) + 0.6521451549\left(\frac{1}{\frac{5}{2}-\frac{3}{2}(0.3399810436)}\right)\right. \\ &\left. +0.347854851\left(\frac{1}{\frac{5}{2}+\frac{3}{2}(0.8611363116)}\right) + 0.347854851\left(\frac{1}{\frac{5}{2}-\frac{3}{2}(0.8611363116)}\right)\right] \\ &= 1.385996821 \end{aligned}$$

The exact solution is

$$\int_0^3 \frac{1}{1+x}dx = \ln|1+x|\Big|_0^3$$
$$= \ln 4 - \ln 1 = \ln 4 = 1.386294361$$

Hence the error is $|1.385996821 - 1.386294361| = 0.00029754$.

(b) Use the forward Euler's method with stepsize $h = 0.1$ to approximate the values of the function y which solves the initial value problem

$$y' = 3x - 2y, \qquad y(0) = 1$$

on the interval $[0, 0.5]$. Then solve the above differential equation and make a table to compare your approximations with the true values.

The Euler method gives $y_{n+1} = y_n + 0.1(3x_n - 2y_n)$. We thus have [3]

$$\begin{aligned}
y_1 &= 1 + 0.1(3(0) - 2(1)) = 0.8\\
y_2 &= 0.8 + 0.1(3(0.1) - 2(0.8)) = 0.67\\
y_3 &= 0.67 + 0.1(3(0.2) - 2(0.67)) = 0.596\\
y_4 &= 0.596 + 0.1(3(0.3) - 2(0.596)) = 0.5668\\
y_5 &= 0.5668 + 0.1(3(0.4) - 2(0.5668)) = 0.57344\\
y_6 &= 0.57344 + 0.1(3(0.5) - 2(0.57344)) = 0.608752
\end{aligned}$$

To solve, we have

$$\begin{aligned}
\frac{dy}{dx} + 2y &= 3x\\
I &= e^{2x}\\
\frac{d}{dx}\left(e^{2x}y\right) &= 3xe^{2x}\\
u &= 3x \qquad\qquad v' = e^{2x}\\
u' &= 3 \qquad\qquad v = \frac{e^{2x}}{2}
\end{aligned}$$

We thus have

$$\begin{aligned}
e^{2x}y &= \frac{3}{2}xe^{2x} - \frac{3}{2}\int e^{2x}dx + c\\
&= \frac{3}{2}xe^{2x} - \frac{3}{4}e^{2x} + c\\
y &= \frac{3}{2}x - \frac{3}{4} + ce^{-2x}\\
y(0) &= -\frac{3}{4} + c = 1\\
c &= \frac{7}{4}\\
y &= \frac{3}{2}x - \frac{3}{4} + \frac{7}{4}e^{-2x}
\end{aligned}$$

The table of values is thus

n	xn	yn	yn+1	y(xn)
0	0	1	0.8	1
1	0.1	0.8	0.67	0.83277882
2	0.2	0.67	0.596	0.72306008
3	0.3	0.596	0.5668	0.66042036
4	0.4	0.5668	0.57344	0.63632569
5	0.5	0.57344	0.608752	0.64378902