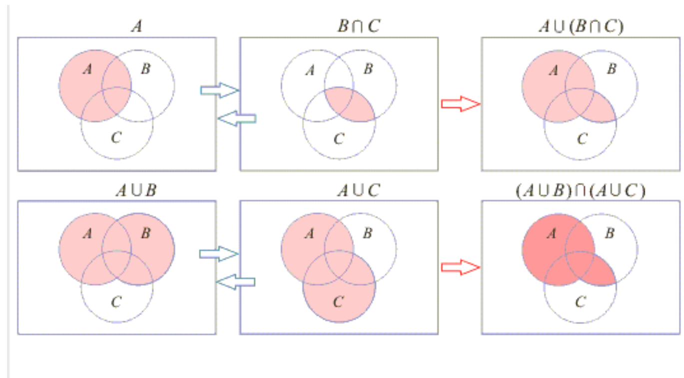
Assignment 1 solution

Question 1



Question 2

Verify $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$$A \cup (B \cap C) = \{x|2 \leq x \leq 10\} \cup \{x|3 \leq x \leq 8\} = \{x|2 \leq x \leq 10\}$$

$$(A \cup B) \cap (A \cup C) = \{x|2 \leq x \leq 20\} \cap \{x|1 \leq x \leq 10\} = \{x|2 \leq x \leq 10\}$$

Verify $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$A \cap (B \cup C) = \{x|2 \leq x \leq 10\} \cap \{x|1 \leq x \leq 20\} = \{x|2 \leq x \leq 10\}$$

$$(A \cap B) \cup (A \cap C) = \{x|3 \leq x \leq 10\} \cup \{x|2 \leq x \leq 8\} = \{x|2 \leq x \leq 10\}$$

Question 3

(a)

$$(x^{\frac{1}{12}} \times x^{-\frac{3}{4}})^{-3} = x + 2$$

$$\rightarrow x^2 - x - 2 = 0$$

$$\rightarrow x_{1,2} = \frac{1 \pm \sqrt{1 - 4(1)(-2)}}{2}$$

$$\rightarrow x_{1,2} = \frac{1 \pm 3}{2}$$

$$\rightarrow x_1 = 2; \quad x_2 = -1$$

(b)

$$
\begin{aligned}
& lnx(x-2) = ln8 \\
& \rightarrow x(x-2) = 8 \\
& \rightarrow x^2 - 2x - 8 = 0 \\
& \rightarrow x_{1,2} = \frac{2 \pm \sqrt{4 - 4(1)(-8)}}{2} \\
& \rightarrow x_{1,2} = \frac{2 \pm 6}{2} \\
& \rightarrow x = 4 \quad \text{since } x \text{ has to be greater than } 2
\end{aligned}
$$

Question 4

$$AB = \begin{bmatrix} 3x+6y \\ 4x+y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Then,

$$
\begin{aligned}
& 3x + 6y = 0 \\
& 4x + y = 2 \\
& x = \frac{4}{7}, \quad y = -\frac{2}{7}
\end{aligned}
$$

Question 5

$$AB = \begin{bmatrix} 7(12)+11(3) & 7(4)+11(6) & 7(5)+11(1) \\ 2(12)+9(3) & 2(4)+9(6) & 2(5)+9(1) \\ 10(12)+6(3) & 10(4)+6(6) & 10(5)+6(1) \end{bmatrix} = \begin{bmatrix} 117 & 94 & 46 \\ 51 & 62 & 19 \\ 138 & 76 & 56 \end{bmatrix}.$$

$$B'A' = \begin{bmatrix} 12 & 3 \\ 4 & 6 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 7 & 2 & 10 \\ 11 & 9 & 6 \end{bmatrix} = \begin{bmatrix} 12(7)+3(11) & 12(2)+3(9) & 12(10)+3(6) \\ 4(7)+6(11) & 4(2)+6(9) & 4(10)+6(6) \\ 5(7)+1(11) & 5(2)+1(9) & 5(10)+1(6) \end{bmatrix} = \begin{bmatrix} 117 & 51 & 138 \\ 94 & 62 & 76 \\ 46 & 19 & 56 \end{bmatrix}$$

$(AB)' = B'A'$.

Question 6

Row 3×(3)+ Row 1 and Row 3×(−1)+ Row 2

$$A = \begin{bmatrix} 7 & -9 & 0 \\ 2 & 6 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Row 2×($\frac{3}{2}$)+ Row 1

$$A = \begin{bmatrix} 10 & 0 & 0 \\ 2 & 6 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Row 1×($-\frac{1}{5}$)+ Row 2

$$A = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Row 2×($\frac{1}{3}$)+ Row 3

$$A = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row 1×($\frac{1}{10}$) and Row 2 ×($\frac{1}{6}$)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A is in echelon form, and $r(A) = 3$, rows in matrix A are *linearly independent*, and A is *square*→ A is non-singular.

Question 7

If matrix A is singular, then its determinant equals to zero. By Laplace expansion,

$$\left|M_{11}\right| = \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} = 6, \quad \left|M_{12}\right| = \begin{vmatrix} 1 & 2 \\ x & 2 \end{vmatrix} = 2 - 2x; \quad \left|M_{13}\right| = \begin{vmatrix} 1 & 3 \\ x & 0 \end{vmatrix} = -3x$$

The corresponding cofactors are:

$$\left|C_{11}\right| = (-1)^{1+1}(6) = 6; \quad \left|C_{12}\right| = (-1)^{1+2}(2-2x) = 2x-2; \quad \left|C_{13}\right| = (-1)^{1+3}(-3x) = -3x$$

The determinant of Matrix A is:

$$\begin{aligned} \left|A\right| &= (1)(6) + (-1)(2x-2) + (2)(-3x) \\ &= 6 + 2 - 2x - 6x = 0 \\ &\rightarrow -8x = -8 \\ &\rightarrow x = 1. \end{aligned}$$

Question 8

(a)

$$\underbrace{\begin{bmatrix} 4 & 3 & -2 \\ 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}}_{b}$$

(b)

$$\begin{aligned} &|M_{11}| = 2; \quad |M_{12}| = 1; \quad |M_{13}| = -6; \\ &|M_{21}| = 3; \quad |M_{22}| = 10; \quad |M_{23}| = -9; \\ &|M_{31}| = 4; \quad |M_{32}| = 2; \quad |M_{33}| = 5 \end{aligned}$$

The corresponding cofactors are:

$$\begin{aligned} &|C_{11}| = 2; \quad |C_{12}| = -1; \quad |C_{13}| = -6; \\ &|C_{21}| = -3; \quad |C_{22}| = 10; \quad |C_{23}| = 9; \\ &|C_{31}| = 4; \quad |C_{32}| = -2; \quad |C_{33}| = 5 \end{aligned}$$

The determinant of matrix A is:

$$|A| = 4(2) + (3)(-1) + (-2)(-6) = 17$$

The adjoint matrix is:

$$Adj A = C' = \begin{bmatrix} 2 & -3 & 4 \\ -1 & 10 & -2 \\ -6 & 9 & 5 \end{bmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{17}\begin{bmatrix} 2 & -3 & 4 \\ -1 & 10 & -2 \\ -6 & 9 & 5 \end{bmatrix}$$

(c)

$$A|I = \begin{bmatrix} 4 & 3 & -2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Row 3×(2)+ Row 1

$$A|I = \begin{bmatrix} 10 & 3 & 0 & 1 & 0 & 2 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Row 2×(−10)+ Row 1

$$A|I = \begin{bmatrix} 0 & -17 & 0 & 1 & -10 & 2 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Row 1×$\frac{2}{17}$+ Row 2

$$A|I = \begin{bmatrix} 0 & -17 & 0 & 1 & -10 & 2 \\ 1 & 0 & 0 & \frac{2}{17} & -\frac{3}{17} & \frac{4}{17} \\ 3 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Row 2×(−3)+ Row 3

$$A|I = \begin{bmatrix} 0 & -17 & 0 & 1 & -10 & 2 \\ 1 & 0 & 0 & \frac{2}{17} & -\frac{3}{17} & \frac{4}{17} \\ 0 & 0 & 1 & -\frac{6}{17} & \frac{9}{17} & \frac{5}{17} \end{bmatrix}$$

Row 1×($-\frac{1}{17}$ and interchange Row 1 and Row 2

$$A|I = \begin{bmatrix} 1 & 0 & 0 & \frac{2}{17} & -\frac{3}{17} & \frac{4}{17} \\ 0 & 1 & 0 & -\frac{1}{17} & \frac{10}{17} & -\frac{2}{17} \\ 0 & 0 & 1 & -\frac{6}{17} & \frac{9}{17} & \frac{5}{17} \end{bmatrix}$$

(d)

$$A_1 = \begin{bmatrix} 1 & 3 & -2 \\ 6 & 2 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$|M_{11}| = 2; \quad |M_{12}| = 6; \quad |M_{13}| = -8$$

$$|C_{11}| = 2; \quad |C_{12}| = -6; \quad |C_{13}| = -8$$

$$|A_1| = 1(2) + 3(-6) + (-2)(-8) = 2 - 18 + 16 = 0$$

$$x^* = \frac{|A_1|}{|A|} = 0$$

$$A_2 = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 6 & 0 \\ 3 & 4 & 1 \end{bmatrix}$$

$$|M_{11}| = 6; \quad |M_{12}| = 1; \quad |M_{13}| = -14$$

$$|C_{11}| = 6; \quad |C_{12}| = -1; \quad |C_{13}| = -14$$

$$|A_1| = 4(6) + 1(-1) + (-2)(-14) = 24 + 1 + 28 = 51$$

$$y^* = \frac{|A_2|}{|A|} = 3$$

$$A_2 = \begin{bmatrix} 4 & 3 & 1 \\ 1 & 2 & 6 \\ 3 & 0 & 4 \end{bmatrix}$$

$$|M_{11}| = 8; \quad |M_{12}| = -14; \quad |M_{13}| = -6$$

$$|C_{11}| = 8; \quad |C_{12}| = 14; \quad |C_{13}| = -6$$

$$|A_1| = 4(8) + 3(14) + (1)(-6) = 32 + 42 - 6 = 68$$

$$z^* = \frac{|A_3|}{|A|} = 4$$

Question 9

After eliminating the quantity variables, the system can be written as

$$a_0 + a_1P_1 + a_2P_2 = b_0 + b_1P_1 + b_2P_2$$

$$c_0 + c_1P_1 + c_2P_2 = d_0 + d_1P_1 + d_2P_2$$

which can be simplified to

$$(a_1 - b_1)P_1 + (a_2 - b_2)P_2 = b_0 - a_0$$

$$(c_1 - d_1)P_1 + (c_2 - d_2)P_2 = d_0 - c_0$$

In matrix form:

$$\underbrace{\begin{bmatrix} a_1 - b_1 & a_2 - b_2 \\ c_1 - d_1 & c_2 - d_2 \end{bmatrix}}_{A} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} b_0 - a_0 \\ d_0 - c_0 \end{bmatrix}$$

We have assumed that the system has unique solution, thus we do not need to check non-singularity of the coefficient-matrix. The determinant of A:

$$|A| = (a_1 - b_1)(c_2 - d_2) - (a_2 - b_2)(c_1 - d_1)$$

By using Cramer's rule,

$$P_1^* = \frac{|A_1|}{|A|}, \quad P_2^* = \frac{|A_2|}{|A|}$$

where

$$A_1 = \begin{bmatrix} b_0 - a_0 & a_2 - b_2 \\ d_0 - c_0 & c_2 - d_2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_1 - b_1 & b_0 - a_0 \\ c_1 - d_1 & d_0 - c_0 \end{bmatrix}$$

Therefore,

$$P_1^* = \frac{(b_0 - a_0)(c_2 - d_2) - (a_2 - b_2)(d_0 - c_0)}{(a_1 - b_1)(c_2 - d_2) - (a_2 - b_2)(c_1 - d_1)}$$
$$P_2^* = \frac{(a_1 - b_1)(d_0 - c_0) - (b_0 - a_0)(c_1 - d_1)}{(a_1 - b_1)(c_2 - d_2) - (a_2 - b_2)(c_1 - d_1)}$$