Solutions

MAT 1339, Fall 2016 Assignment 1

Due Sep 29th 11:30 am.

Late assignments will **NOT** be accepted. Assignments are to be handed in directly to the professor at the beginning of lecture.

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Student Name ______________________________ Student Number ______________________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ______________________________

QUESTION 1. Let $f(x) = \dfrac{2}{x+3}$.

a) Determine the average rate of change of f over the interval $[0, 1]$.

$$\frac{\Delta f}{\Delta x} = \frac{f(1) - f(0)}{1-0} = \frac{\frac{2}{1+3} - \frac{2}{0+3}}{1} = \frac{2}{4} - \frac{2}{3} = \frac{6-8}{12} = -\frac{1}{6}$$

b) Determine the difference equation at the point $x = 0$ and simplify it.

$$\frac{f(0+h) - f(0)}{h} = \frac{\frac{2}{0+h+3} - \frac{2}{0+3}}{h} = \frac{1}{h}\left[\frac{2}{h+3} - \frac{2}{3}\right] = \frac{1}{h}\left[\frac{6-2(h+3)}{3(h+3)}\right] = \frac{-2\cancel{h}}{3\cancel{h}(h+3)} = \frac{-2}{3(h+3)}$$

c) Estimate the slope of the tangent line at the point $x = 0$ with $h = 1/10$.

Put $h = 1/10$ into the simplified answer from part (b):

$$\frac{-2}{3(1/10+3)} = \frac{-2}{3 \cdot 3.1} = \frac{-2}{9.3} \approx -0.2151$$

d) Use First Principles to find the exact slope of the tangent line at the point $x = 0$.

Again, using what we found in part (b):

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{-2}{3(h+3)} = \frac{-2}{3\left(\lim_{h \to 0} h + 3\right)}$$

$$= \frac{-2}{3(0+3)}$$

$$= -2/9 \approx -0.2222$$

QUESTION 2. Evaluate the following limits:

a) $\lim_{x \to -2} \frac{x+2}{x^2+1} = \frac{\lim_{x \to -2} x + 2}{\left(\lim_{x \to -2} x\right)^2 + 1} = \frac{-2+2}{(-2)^2+1} = \frac{0}{5} = 0$

Using the rules from class

b) $\lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 3}$ ⊛ Notice simply putting $x=3$ into this gives $\frac{0}{0}$. Factor!

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(2x+1)}{\cancel{(x-3)}}$$

$$= \lim_{x \to 3} 2x + 1 = 2\left(\lim_{x \to 3} x\right) + 1 = 2(3) + 1 = 7$$

c) $\lim_{x \to 0} \frac{5 - \sqrt{25 - x}}{x}$ ⊛ Notice simply putting $x=0$ into this again gives $\frac{0}{0}$. Multiply by conjugate!

$$= \lim_{x \to 0} \frac{(5 - \sqrt{25-x})(5 + \sqrt{25-x})}{x(5 + \sqrt{25-x})} = \lim_{x \to 0} \frac{25 - (25 - x)}{x(5 + \sqrt{25-x})}$$

$$= \lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}(5 + \sqrt{25-x})}$$ ← No problem Now!

$$= \frac{1}{10}$$

d) $\lim_{x \to +\infty} \frac{5x^2 + 10x - 3}{-2x^2 + 12} = \lim_{x\to\infty} \frac{\cancel{x^2}(5 + {}^{10}/_{x} - {}^{3}/_{x^2})}{\cancel{x^2}(-2 + {}^{12}/_{x^2})}$

$= \lim_{x\to\infty} \frac{5 + {}^{10}/_{x} - {}^{3}/_{x^2}}{-2 + {}^{12}/_{x^2}}$

$= \frac{\lim_{x\to\infty} 5 + \lim_{x\to\infty} {}^{10}/_{x} - \lim_{x\to\infty} {}^{3}/_{x^2}}{\lim_{x\to\infty} -2 + \lim_{x\to\infty} {}^{12}/_{x^2}} = \frac{5+0-0}{-2+0}$

$= -5/2$

e) $\lim_{x \to +\infty} \frac{101}{x + 1001} =$

$= \lim_{x\to\infty} \frac{\cancel{x}\left(\frac{101}{x}\right)}{\cancel{x}\left(1 + \frac{1001}{x}\right)} = \frac{\lim_{x\to\infty} \frac{101}{x}}{\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{1001}{x}}$

$= \frac{0}{1+0} = 0$

f) $\lim_{x \to -2} \frac{|x + 2|}{x^2 - 4} =$

Recall that: $|x+2| = \begin{cases} -(x+2) & \text{if } x+2 < 0 \\ x+2 & \text{if } x+2 \geq 0 \end{cases}$

$= \begin{cases} -x-2 & \text{if } x < -2 \\ x+2 & \text{if } x \geq -2 \end{cases}$

Therefore, this limit is going to look different when coming from the left or the right:

i) $\lim_{x\to -2^-} \frac{|x+2|}{x^2-4} = \lim_{x\to -2^-} \frac{-(x+2)}{x^2-4} = \lim_{x\to -2^-} \frac{-1\cancel{(x+2)}}{\cancel{(x+2)}(x-2)} = \frac{-1}{-4} = \frac{1}{4}$

ii) $\lim_{x\to -2^+} \frac{|x-2|}{x^2-4} = \lim_{x\to -2^+} \frac{x+2}{x^2-4} = \lim_{x\to -2^+} \frac{\cancel{(x+2)}}{\cancel{(x+2)}(x-2)} = \frac{1}{-4} = \frac{-1}{4}$

But, the limits from the left and the right are not equal! So the limit doesn't exist!

QUESTION 3. Find the values of a and b in such a way that f is a continuous function.

$$f(x) = \begin{cases} \sqrt{x+a}+1 & \text{if } x \le 0, \\ 2x^2+3 & \text{if } 0 < x \le 1, \\ (x+b)^2+1 & \text{if } x > 1. \end{cases}$$

We check at the points where the function is "patched" together: $x=0,1$

$\underline{x=0/}$ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sqrt{x+a}+1 = \sqrt{a}+1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2x^2+3 = 3$

To be continuous, we need these to be equal:

$\Rightarrow \sqrt{a}+1 = 3$

$\Rightarrow \sqrt{a} = 2$

$\Rightarrow \boxed{a=4}$

$\underline{x=1/}$ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x^2+3 = 2(1)^2+3 = 5$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+b)^2+1 = (1+b)^2+1$

Again, need these to be equal:

$\Rightarrow (1+b)^2+1 = 5$

$(1+b)^2 = 4$

When taking the square root we have two possibilities!

$\Rightarrow 1+b=2$ or $1+b=-2$ ✱

$\Rightarrow b=1$ or $b=-3$

Two choices: $\boxed{b=-3,1}$

QUESTION 4. Find the set of points that the following function is continuous at them.

$$f(x) = \begin{cases} \dfrac{2x-1}{x+3} & \text{if } x < 0, \\ \\ \dfrac{\sqrt{x+4}-2}{2x^2-3x} & \text{if } x > 0 \\ \\ \frac{-1}{3} & \text{if } x = 0 \end{cases}$$

Inspect $f(x)$ on the two half intervals first:

For $x<0$: $f(x) = \dfrac{2x-1}{x+3}$

Notice that at $x=-3$ the denominator becomes 0, therefore, since the numerator doesn't vanish also, we find that $x=-3$ is not in the domain & $f(x)$ is not continuous at $x=-3$. Hence, on $x<0$,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{2x-1}{x+3} = \frac{2(0)-1}{0+3} = -\tfrac{1}{3}$$

For $x>0$: $f(x) = \dfrac{\sqrt{x+4}-2}{2x^2-3x}$

Again, denominator vanishes at $x=0, \tfrac{3}{2}$. Now only $x=\tfrac{3}{2}$ is positive and the numerator doesn't vanish here, so $f(x)$ is not continuous at $x=\tfrac{3}{2}$. Notice that at $x=0$ the quotient becomes $\tfrac{0}{0}$. So,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sqrt{x+4}-2}{2x^2-3x}$$ ← Multiply by conjugate & factor bottom

$$= \lim_{x \to 0^+} \frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(2x-3)(\sqrt{x+4}+2)}$$

$$= \lim_{x \to 0^+} \frac{\cancel{x}+\cancel{4}-\cancel{4}}{\cancel{x}(2x-3)(\sqrt{x+4}+2)}$$

$$= \lim_{x \to 0^+} \frac{1}{(2x-3)(\sqrt{x+4}+2)}$$ ← No more problems at $x=0$!

$$= \frac{1}{(2(0)-3)(\sqrt{0+4}+2)}$$

$$= \frac{-1}{12}$$

Now, we see that $\lim_{x \to 0^-} f(x) = -\frac{1}{3} = f(0)$

but $\lim_{x \to 0^+} f(x) \neq -1/3$, so $f(x)$ is not

continuous at $x=0$.

Therefore, $f(x)$ is continuous on:

$$(-\infty, -3) \cup (-3, 0) \cup (0, 3/2) \cup (3/2, \infty)$$

or $\mathbb{R} \setminus \{-3, 0, 3/2\}$