Solutions

School of Mathematics and Statistics
Carleton University
Math. 2004A, Fall 2014
TEST 1

STUDIO 56 calculator ONLY permitted, 1 or more blank sheets permitted for roughs

Print Name :

Student Number:

Tutorial Section (A1, A4, ...):

PART I: Multiple Choice Questions
(Choose and CIRCLE only ONE answer - No part marks here.)

1. **[2 marks]** What value of x will make the two vectors $(1, 2, 0)$ and $(-2, -4, x)$ parallel?
(a) $x = 0$, (b) $x = -1$, (c) $x = +1$, (d) $x = 1/2$.
{ Set cross product $= \vec{0}$

2. **[2 marks]** Find the equation of the line through $(1, 3, 2)$ that is parallel to the vector $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.
(a) $x = 2 + t, y = 4 + 3t, z = 5 + 2t$, (b) $x = 2 - t, y = 4 + 3t, z = 5 - 2t$, (c) $x = 1 + 2t, y = 3 + 4t, z = 2 + 5t$, (d) $x = 1 + 2t, y = 1 - 3t, z = 1 + 2t$.
(clear).

3. **[2 marks]** Let C denote the cardioid defined by the equation $r = 2 + \sin\theta$, $0 \le \theta \le 2\pi$, in plane polar coordinates. Find the maximum distance of a point P on C from the origin.
(a) 2, (b) 3, (c) 1, (d) 4.
(Set $\frac{dr}{d\theta} = 0 \Rightarrow r = \pi/2$).

4. **[2 marks]** Convert the equation of the surface $x^2 + y^2 = 2z$ into cylindrical coordinates.
(a) $z = 2r$, (b) $r = 2z$, (c) $r - 2z + 1 = 0$, (d) $r^2 = 2z$.
($\because x^2+y^2 = r^2$).

5. **[2 marks]** The planes $x + 2y + z = 1$ and $2x - 3y + 4z = 3$ are orthogonal.
(a) TRUE, (b) FALSE.
($\because$ their normal vectors are orthogonal).

PART II: Show all work here and give details.
No additional pages will be accepted

6. **[10 marks]** Find the equation of the plane passing through the point $(3, -4, 5)$ and containing the line

$$\frac{x-2}{2} = \frac{y+1}{-3} = \frac{z+3}{5}.$$

Let $P(3,-4,5)$.

The line may be written parametrically in the form

(1) $\rightarrow$ $\mathcal{L}$: $x = 2+2t$, $y = -1-3t$, $z = -3+5t$, $t \in \mathbb{R}$.

(2) { All we need is 2 points Q, R on $\mathcal{L}$ that generate 2 vectors $\vec{PQ}, \vec{PR}$ such that $\vec{n} = \vec{PQ} \times \vec{PR}$ is orthogonal to the plane.

e.g. let $Q = (2,-1,-3)$ (set $t=0$)
$R = (0,2,-8)$ (set $t=-1$)
} any other two points would be just as good.

(2) Then $\vec{PQ} = (-1,3,-8)$
(2) $\vec{PR} = (-3,6,-13)$.

So $\vec{n} = \begin{vmatrix} i & j & k \\ -1 & 3 & -8 \\ -3 & 6 & -13 \end{vmatrix} = 9\vec{i} + 11\vec{j} + 3\vec{k}$ or $(9,11,3)$ is normal to the plane.

i.e. $\boxed{\vec{n} = (9,11,3)}$ — (1)

$\therefore$ Equation of plane is $9x + 11y + 3z = D$

& now use any one point on it to get "D". Say $x = 3$, $y = -4$, $z = 5$ $\Rightarrow$ $\boxed{D = -2}$ $\longleftarrow$ (1)

$\therefore$ $\boxed{9x + 11y + 3z = -2}$ $\Leftarrow$ (1)

7. **[10 marks]** Find the area of the region bounded by the curve $r = 5\sin\theta$ where $0 \le \theta \le \pi$.

Observe that this is a circle of radius $5/2$ centered at $(0, 5/2)$

$\therefore$ its area $= \pi\left(\frac{5}{2}\right)^2 = \boxed{\frac{25\pi}{4}}$. $\leftarrow$ 5 3 2

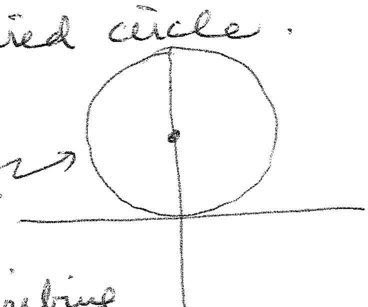
Why? $r = 5\sin\theta \Rightarrow r^2 = 5r\sin\theta$

or $x^2 + y^2 = 5y$

$\therefore$ $x^2 + y^2 - 5y = 0$. + completing the square.

$(x-0)^2 + \left(y - \frac{5}{2}\right)^2 = \left(\frac{5}{2}\right)^2$ is the desired circle.

OR



$r = 5\sin\theta$.

Observe that $0 \le \theta \le \pi$ and not $0 \le \theta \le 2\pi$ as then we would be describing the circle "twice" or "doubling" the area.

2

$\therefore$ Area $= \frac{1}{2}\int_0^{\pi} r^2\, d\theta = \frac{1}{2}\int_0^{\pi} 25\sin^2\theta\, d\theta$

3

$= \frac{1}{2} \cdot 25 \int_0^{\pi} \frac{(1-\cos 2\theta)}{2}\, d\theta = \underbrace{\frac{25}{4}\int_0^{\pi} (1-\cos 2\theta)\, d\theta}_{2}$

$= \frac{25\pi}{4} - \frac{25}{4}\int_0^{\pi} \cos 2\theta\, d\theta$

$= \frac{25\pi}{4} - \frac{25}{4}\left(\frac{\sin 2\theta}{2}\right)\Big|_0^{\pi}$

0

$= \frac{25\pi}{4}$ $\leftarrow$ 3