MATH 1005C Test 2 Solutions

1. Solve the following **homogeneous** second-order DEs. If initial values are provided, use them to solve for the constants.

   (a) [5 marks]

$$4y'' - 4y' + y = 0, \qquad \begin{cases} y(0) = 3 \\ y'(0) = 1 \end{cases}$$

Indicial equation: $4r^2 - 4r + 1 = 0$, which gives $r_1 = r_2 = \frac{1}{2}$. We have Case 2, real and equal roots, with the following solution:

$$y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}$$

To solve for the constants, we need the derivative too:

$$\begin{aligned} y' &= \frac{1}{2}c_1 e^{\frac{1}{2}x} + \frac{1}{2}c_2 x e^{\frac{1}{2}x} + c_2 e^{\frac{1}{2}x} \\ &= c_1(\frac{1}{2}e^{\frac{1}{2}x}) + c_2(\frac{1}{2}xe^{\frac{1}{2}x} + e^{\frac{1}{2}x}) \end{aligned}$$

Now we apply the initial conditions:

$$3 = y(0) = c_1$$

$$1 = y'(0) = \frac{1}{2}c_1 + c_2$$

So $c_1 = 3$, $c_2 = 1 - c_1 = -\frac{1}{2}$, and the final answer is

$$y = 3e^{\frac{1}{2}x} - \frac{1}{2}xe^{\frac{1}{2}x}$$

   (b) [4 marks]

$$3x^2y'' + 14xy' - 4y = 0 \qquad (x \neq 0)$$

Indicial equation: $3r(r-1) + 14r - 4 = 3r^2 + 11r - 4 = 0$ (don't forget the slight difference for Cauchy-Euler equations!) This factors as $(3r-1)(r+4) = 0$, so the roots are $r_1 = \frac{1}{3}$, $r_2 = -4$. It's Case 1, real and distinct roots, and our solution is

$$y = c_1 x^{\frac{1}{3}} + c_2 x^{-4}.$$

The question already specifies $x \neq 0$, so we don't need to repeat that. We also do *not* need absolute value bars for these particular powers – $\frac{1}{3}$ has an odd denominator, so it's defined for all real numbers.

(That said, it would be equally valid to write

$$y = k_1|x|^{\frac{1}{3}} + k_2x^{-4},$$

which is equivalent to setting $k_1 = -c_1$ for negative x and $k_1 = c_1$ for positive x. Normally this would cause a cusp at $x = 0$, but that point isn't in our domain at all. Thus we can match any $x < 0$ solution with any $x > 0$ solution and not worry about the join.

Cauchy-Euler equations always have issues at $x = 0$, even if the solution turns out to be continuous there. This is because the DE itself degenerates into $y = 0$ when 0 is plugged in. A point where this happens to a DE is called a *singular point*. We'll be studying them later.

Incidentally, this is the same phenomenon I alluded to in the Test 1 answers when I said there was a reason question 3 had solutions that blew up at $x = 0$.)

(c) [4 marks]

$$y'' + 8y' + 17y = 0$$

Indicial equation: $r^2 + 8r + 17 = 0$. Completing the square or using the quadratic formula gives $r = -4 \pm i$. We have Case 3, complex conjugate roots, with $\alpha = -4$ and $\beta = 1$. Thus the solution is

$$y = e^{-4x}(c_1 \cos x + c_2 \sin x).$$

2. Solve the following **inhomogeneous** second-order DEs by any method.

(a) [6 marks]

$$y'' + 9y = 6\cos 3x$$

First we need the homogenous solution. Indicial equation: $r^2 + 9 = 0$, which gives $r = \pm 3i$. Case 3 with $\alpha = 0$ and $\beta = 3$:

$$y_h = c_1 \cos 3x + c_2 \sin 3x$$

As promised in class, a particular solution can be found by either of the two methods we've learned.

- [Undetermined coefficients]

  With $6\cos 3x$ on the right side, we would normally seek a solution of the form

  $$y_p = A\cos 3x + B\sin 3x$$

Unfortunately, this is a solution of the homogenous equation, so we have to throw in an x (left outside to make differentiation simpler):

$$y_p = x(A\cos 3x + B\sin 3x)$$

Now

$$\begin{aligned}
y_p' &= (A\cos 3x + B\sin 3x) + x(-3A\sin 3x + 3B\cos 3x) \\
y_p'' &= (-3A\sin 3x + 3B\cos 3x) + (-3A\sin 3x + 3B\cos 3x) + x(-9A\cos 3x - 9B\sin 3x) \\
&= (-6A\sin 3x + 6B\cos 3x) + x(-9A\cos 3x - 9B\sin 3x)
\end{aligned}$$

Plug into the original DE:

$$y_p'' + 9y = 6\cos 3x$$

$$(-6A\sin 3x + 6B\cos 3x) + \cancel{x(-9A\cos 3x - 9B\sin 3x)} + \cancel{9x(A\cos 3x + B\sin 3x)} = 6\cos 3x$$

$$-6A\sin 3x + 6B\cos 3x = 6\cos 3x$$

So $A = 0$, $B = 1$, and $y_p = x\sin 3x$.

(The large cancellation above is typical of questions where we try a y_p that looks like xy_h. This is another example of *reduction of order*, the technique we used to get a y_2 for some homogeneous second-order DEs.)

- [Variation of parameters]

  Let $y_1 = \cos 3x$, $y_2 = \sin 3x$. Then

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} \\
&= 3\cos^2 3x + 3\sin^2 3x \\
&= 3.
\end{aligned}$$

  Thus a particular solution $y_p = u_1y_1 + u_2y_2$ will be given by

$$\begin{aligned}
u_1 &= \int \frac{-y_2 f}{W}\ dx \\
&= \int \frac{-6\sin 3x\cos 3x}{3}\ dx \\
&= -\int 2\sin 3x\cos 3x\ dx \\
&= -\int \sin 6x\ dx \\
&= \frac{1}{6}\cos 6x
\end{aligned}$$

(Hey, why doesn't that look like anything in the undetermined-coefficients solution? Don't panic...)

$$\begin{aligned} u_2 &= \int \frac{y_1 f}{W}\ dx \\ &= \int \frac{6\cos 3x \cos 3x}{3}\ dx \\ &= \int 2\cos^2 3x\ dx \\ &= \int (2\cos^2 3x - 1) + 1\ dx \\ &= \int \cos 6x + 1\ dx \\ &= \frac{1}{6}\sin 6x + x \end{aligned}$$

Putting this together,

$$\begin{aligned} y_p &= (\frac{1}{6}\cos 6x)\cos 3x + (\frac{1}{6}\sin 6x + x)\sin 3x \\ &= \frac{1}{6}(\cos 6x \cos 3x - \sin 6x \sin 3x) + x \sin 3x \end{aligned}$$

If you went ahead and used this as y_p, you didn't lose any marks – but it does simplify. Think back to your addition/subtraction formulas:

$$\cos 6x \cos 3x - \sin 6x \sin 3x = \cos(6x - 3x) = \cos 3x$$

So y_p can be written as $\frac{1}{6}\cos 3x + x\sin 3x$ – and the first term is a solution of the homogeneous equation, so we can remove it entirely and get the equally valid $y_p = x\sin 3x$, giving us the same solution as by undetermined coefficients.

By either method, the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 \cos 3x + c_2 \sin 3x + x \sin 3x. \end{aligned}$$

(b) [6 marks]

$$y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$$

First we need the homogenous solution. Indicial equation: $r^2 + 2r - 8 = 0$, which gives $r_1 = -4$, $r_2 = 2$. Case 1, real and distinct roots, with the general solution

$$y_h = c_1 e^{-4x} + c_2 e^{2x}.$$

This time $f(x)$ consists of two incompatible exponential terms. Recalling that DEs are linear, we can find a y_p for each one and add them, or we can do both at once.

- [Undetermined coefficients]

  For this method, let's handle each term of f separately. A y_p for $2e^{-2x}$ will look like

$$y_p = Ae^{-2x}, \quad y_p' = -2Ae^{-2x}, \quad y_p'' = 4Ae^{-2x}$$

  so

$$\begin{aligned}(4Ae^{-2x}) + 2(-2Ae^{-2x}) - 8(Ae^{-2x}) &= 2e^{-2x} \\ -8A &= 2 \\ A &= -\frac{1}{4}\end{aligned}$$

  and our y_p for the first part is $-\frac{1}{4}e^{-2x}$. On the other hand, a y_p for $-e^{-x}$ will look like

$$y_p = Be^{-x}, \quad y_p' = -Be^{-x}, \quad y_p'' = Be^{-x}$$

  and so

$$\begin{aligned}(Be^{-x}) + 2(-Be^{-x}) - 8(Be^{-x}) &= -e^{-x} \\ -9B &= -1 \\ B &= \frac{1}{9}\end{aligned}$$

  giving us $y_p = \frac{1}{9}e^{-x}$ for the second part. Therefore, our y_p for the whole DE is

$$y_p = -\frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x}.$$

- [Variation of parameters]

  We have $y_1 = e^{-4x}$, $y_2 = e^{2x}$ from the homogeneous solution. The Wronskian is

$$\begin{aligned}W(y_1, y_2) &= \begin{vmatrix} e^{-4x} & e^{2x} \\ -4e^{-4x} & 2e^{2x} \end{vmatrix} \\ &= 2e^{-2x} + 4e^{-2x} \\ &= 6e^{-2x}.\end{aligned}$$

  This time, let's take $f(x)$ in one piece. We have

$$\begin{aligned}u_1 &= \int \frac{-y_2 f}{W}\, dx \\ &= \int \frac{-e^{2x}(2e^{-2x} - e^{-x})}{6e^{-2x}}\, dx \\ &= \int \frac{-2 + e^{x}}{6e^{-2x}}\, dx \\ &= \frac{1}{6}\int -2e^{2x} + e^{3x}\, dx \\ &= \frac{1}{6}(-e^{2x} + \frac{1}{3}e^{3x})\end{aligned}$$

and

$$
\begin{aligned}
u_2 &= \int \frac{y_1 f}{W}\, dx \\
&= \int \frac{e^{-4x}(2e^{-2x} - e^{-x})}{6e^{-2x}}\, dx \\
&= \int \frac{2e^{-6x} - e^{-5x}}{6e^{-2x}}\, dx \\
&= \frac{1}{6}\int 2e^{-4x} - e^{-3x}\, dx \\
&= \frac{1}{6}(-\frac{1}{2}e^{-4x} + \frac{1}{3}e^{-3x})
\end{aligned}
$$

So our particular solution is

$$
\begin{aligned}
y_p &= \frac{1}{6}(-e^{2x} + \frac{1}{3}e^{3x})e^{-4x} + \frac{1}{6}(-\frac{1}{2}e^{-4x} + \frac{1}{3}e^{-3x})e^{2x} \\
&= \frac{1}{6}(-e^{-2x} + \frac{1}{3}e^{-x}) + \frac{1}{6}(-\frac{1}{2}e^{-2x} + \frac{1}{3}e^{-x}) \\
&= (-\frac{1}{6} - \frac{1}{12})e^{-2x} + (\frac{1}{18} + \frac{1}{18})e^{-x} \\
&= -\frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x},
\end{aligned}
$$

just as we found by undetermined coefficients.

Either way, the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1 e^{-4x} + c_2 e^{2x} - \frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x}.
\end{aligned}
$$

BONUS [3 marks]: Solve by reducing to first order.

$$\frac{y''}{y'} = x + \frac{1}{x}$$

Since y is missing, we make the substitution $u = y'$, $u' = y''$ to get

$$\frac{u'}{u} = x + \frac{1}{x}$$

$$\int \frac{du}{u} = \int x + \frac{1}{x}\, dx$$

$$\ln|u| = \frac{1}{2}x^2 + \ln|x| + k_1 \quad (k_1 \in \mathbb{R})$$

$$|u| = k_2|x|e^{\frac{1}{2}x^2} \quad (k_2 > 0)$$
$$u = k|x|e^{\frac{1}{2}x^2} \quad (k \neq 0)$$

Now we have to integrate to get back to y. The absolute value bars would require two cases, but we can drop them – though not because of the arbitrary constant. The real reason is similar to question 1(b) above: $kxe^{\frac{1}{2}x^2}$ is just the left half of $-k|x|e^{\frac{1}{2}x^2}$ and the right half of $k|x|e^{\frac{1}{2}x^2}$, and they even join up nicely at $(0,0)$, though we don't care since $x = 0$ isn't allowed anyway (see original DE).

Replacing k with c_1 (this is why I used ks originally), we have

$$y = \int c_1 x e^{\frac{1}{2}x^2}\, dx$$

$$y = c_1 e^{\frac{1}{2}x^2} + c_2$$

where $c_1 \neq 0$ but c_2 is unrestricted. (The integral is done by subbing for $\frac{1}{2}x^2$. You should be prepared for "silent" substitutions like this, since you'll see them increasingly often.)

If you kept the $|x|$ and integrated like it was x, or thought the arbitrary constant could absorb the absolute value bars, you didn't lose marks – it's a subtle point and this is just a bonus question. You did lose a mark if you left out absolute values entirely, though.