uOttawa

1. [2 points] Find the area between $f(x) = 2x$ and $g(x) = x^2$ for $0 \le x \le 2$.

Solution: We remark that the formula for finding the area enclosed by two functions $f(x)$ and $g(x)$, and $x = a$, $x = b$ is

$$\int_a^b |f(x) - g(x)|\, dx.$$

By the above remark, the area is

$$\int_0^2 |x^2 - 2x|\, dx.$$

But $0 \le x \le 2$, then by multiplying x in the inequality we get $0 \le x^2 \le 2x$. Therefore in the interval $[0, 2]$ we have (1 point)

$$|x^2 - 2x| = 2x - x^2.$$

Hence the area is (1 point)

$$\int_0^2 (2x - x^2)\, dx = x^2 - \frac{1}{3}x^3\Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}.$$

2. [3 points] Solve the separable differential equation

$$\frac{dy}{dt} = \frac{16te^{t^2+1}}{y}, \tag{1}$$

with the **initial condition** $y(0) = 2\sqrt{e}$.

Solution: This is a Separable Differential equation. We have (1 point)

$$\int y dy = \int 16te^{t^2+1} dt.$$

Therefore (using the power rule and a SUB: $u = t^2 + 1$, so $\frac{du}{dt} = 2t$) we obtain (1 point)

$$\frac{y^2}{2} = 16 \int e^u \frac{du}{2} = 8 \int e^u du = 8e^u + c = 8e^{t^2+1} + c,$$

where c is a number. The initial condition says: $2\sqrt{e} = y(0)$, so $4e = y(0)^2$, hence

$$2e = \frac{(y(0))^2}{2} = 8e^{0+1} + c.$$

Thus $c = 2e - 8e = -6e$. From $\frac{y^2}{2} = 8e^{t^2+1} - 6e$, one has $y = \pm\sqrt{16e^{t^2+1} - 12e}$. Since the Initial Condition is Positive, it follows that the solution is ONLY (1 point):

$$y = \sqrt{16e^{t^2+1} - 12e}.$$

3. [4 points] Find the indefinite integral

$$\int \frac{2x^3 - 18x^2 + 39x + 1}{x^2 - 9x + 20}\, dx. \tag{2}$$

Solution: By LONG DIVISION one has that (1 point)

$$\frac{2x^3 - 18x^2 + 39x + 1}{x^2 - 9x + 20} = 2x + \frac{-x + 1}{x^2 - 9x + 20},$$

which can be written as (1 point):

$$\frac{2x^3 - 18x^2 + 39x + 1}{x^2 - 9x + 20} = 2x + \frac{-x + 1}{(x-4)(x-5)}.$$

Therefore

$$\frac{-x + 1}{(x-4)(x-5)} = \frac{A}{x-4} + \frac{B}{x-5},$$

and so

$$-x + 1 = A(x-5) + B(x-4) = x(A+B) - 5A - 4B.$$

Hence $A + B = -1$, and $-5A - 4B = 1$. Thus $A = -1 - B$ and $-5(-1 - B) - 4B = 1$. We get $B = -4$ and $A = 3$ (1 point).

From this we obtain (1 point)

$$\int \frac{2x^3 - 18x^2 + 39x + 1}{x^2 - 9x + 20}\, dx = x^2 + 3\ln|x-4| - 4\ln|x-5| + c.$$

4. [3 points] For the following improper integral, determine whether it converges, and determine its value if it does.

$$\int_2^\infty \frac{3}{4+8x^2}\,dx. \tag{3}$$

Solution: By the very definition of an improper integral one has:

$$I = \lim_{t\to\infty}\int_2^t \frac{3}{4+8x^2}dx = \frac{3}{4}\lim_{t\to\infty}\int_2^t \frac{1}{1+2x^2}dx = \frac{3}{4}\lim_{t\to\infty}\int_2^t \frac{1}{1+(\sqrt{2}x)^2}dx.$$

Now use a SUB: $u = \sqrt{2}x$, and notice that $\frac{du}{dx} = \sqrt{2}$, hence $\frac{du}{\sqrt{2}} = dx$. Our integral becomes: (2 points)

$$\begin{aligned}
I &= \frac{3}{4}\lim_{t\to\infty}\int_{2\sqrt{2}}^{t\sqrt{2}} \frac{1}{1+(u)^2}\frac{du}{\sqrt{2}} \\
&= \frac{3}{4\sqrt{2}}\lim_{t\to\infty}\arctan(u)|_{2\sqrt{2}}^{t\sqrt{2}} \\
&= \frac{3}{4\sqrt{2}}\lim_{t\to\infty}\{\arctan(t\sqrt{2}) - \arctan(2\sqrt{2})\} \\
&= \frac{3}{4\sqrt{2}}\{\frac{\pi}{2} - \arctan(2\sqrt{2})\} \approx 0.18 \in \mathbf{R}.
\end{aligned} \tag{4}$$

Thus it is convergent (1 point).

5. [4 points] When a murder is committed, the body, originally at 37°C, cools according to Newton's Law of Cooling. Suppose that after two hours the temperature is 35°C and that the temperature of the surrounding air is a constant 20°C. If the body is found at 4 pm at a temperature of 30°C, when was the murder committed?

Solution: Let H be the temperature of the body. According to the Newton's law of cooling, the temperature of a warm object decreases at a rate proportional to the difference between its temperature and that of its surrounding. Namely

$$\frac{dH}{dt} = \alpha(A - H).$$

Hence by solving the differential equation we obtain the following equation (1 point)

$$H(t) = A - (A - H_0)e^{-\alpha t},$$

where $A = 20$, $H_0 = 37$ (1 point). Therefore

$$H(t) = 20 + 17e^{-\alpha t}.$$

We know that after two hours the temperature is 35°C, therefore (1 point)

$$35 = H(2) = 20 + 17e^{-2\alpha} \Longrightarrow \frac{15}{17} = e^{-2\alpha} \Longrightarrow \alpha = -\frac{1}{2}\ln\left(\frac{15}{17}\right) \approx 0.06.$$

So we get the formula

$$H(t) = 20 + 17e^{-0.06t}.$$

To solve the problem we need to know when the temperature reaches 30°C. The idea is to substitute $H = 30$ in the equation and then find t.

$$30 = 20 + 17e^{-0.06t} \Longrightarrow \frac{10}{17} = e^{-0.06t} \Longrightarrow t = \frac{-1}{0.06}\ln\left(\frac{10}{17}\right) \approx 8.84.$$

Thus the murder must have been committed about 8.84 hours before 4 pm. Since 8.4 hours= 8 hours 50 minutes, the murder was committed at about 7:10 am (1 point).

6. [4 points] There is a model of the growth of a limited population called the *Gompertz function*, which is a solution of the differential equation

$$\frac{dP}{dt} = c \ln\left(\frac{M}{P}\right) P.$$

Consider a moose population for which $c = 0.05$ and $M = 1000$.

Part A: [1 point] Find all biologically reasonable equilibrium points.

Solution: We need to find those functions P such that $\frac{dP}{dt} = 0$. Therefore we have the equation

$$0.05 \ln\left(\frac{1000}{P}\right) P = 0 \Rightarrow \ln\left(\frac{1000}{P}\right) P = 0 \Rightarrow P = 0, \qquad P = 1000.$$

Hence biologically reasonable equilibrium points is $P = 1000$.

Part B: [1 point] Determine the stability of each biologically reasonable equilibrium.

Solution: Set

$$f(P) = 0.05 \ln\left(\frac{1000}{P}\right) P.$$

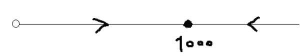
Then we have

$$f'(P) = 0.05\left(\ln\left(\frac{1000}{P}\right) - 1\right).$$

But $f'(1000) = -0.05 < 0$ and so it is a stable equilibrium.

Part C: [1 point] Draw a phase-line diagram.

Solution:



Part D: [1 point] If ten moose are introduced into the environment, how many moose will there be eventually?

Solution: 1000.