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/9/2016

Recall: A <u>Vector space</u> is a set V of objects ('vectors') equipped with 2 operation, addition and multiplication by scalars such that the following [10] conditions (axioms) called

"closure" {
- For every $u, v \in V$, $u+v \in V$ ("closure under addition")
- For every $v \in V$ and every $k \in \mathbb{R}$, $k.v \in V$ ("closure under multp by scalars")

existence {
- There is an object in V, denoted 0 ('zero'), s.t $v+0=v$ for all $v \in V$ (s.t = such that)
- For every $v \in V$ there is an object denoted $-v$, in V s.t. $v+(-v)=0$ ("existence of negative")

arithmetic {
- $u+v=v+u$
- $u+(v+w)=(u+v)+w$
- $k(u+v)=ku+kv$
- $(k+l)u=ku+lu$
- $k.(lu)=(kl)u$
- $1u=u$

for all $u, v, w \in V$ and $k, l \in \mathbb{R}$

<u>Examples</u>

$\mathbb{R}^2$, with usual vector operation
$\mathbb{R}^3$, " " "
$\mathbb{R}^n$, " " "

$V=\{cat\}$: $cat+cat=cat$
for all $k \in \mathbb{R}$ $k.cat=cat$ ("closed under add & multp by scalars)

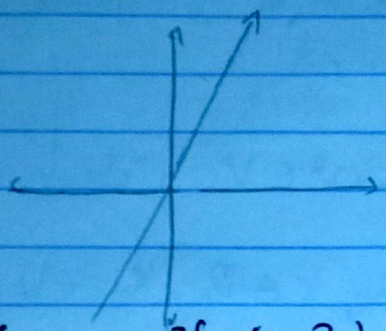
<u>Existence</u>: The object cat is a zero ✓
$cat+\overset{0}{cat}=cat$

Negative: $-cat=cat$: $\to$ $cat+(cat)=cat$ ✓

6 <u>Arithmetic axioms</u> ✓

($V=\{0\}$
$0+0=0$
$k.0=0$
for all k

Example: $V = \{(x, 2x) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$



Use the standard operation inherited $\mathbb{R}$

Closure: If $(x_1, 2x)$ & $(y, 2y) \in V$, $\quad x, y \in \mathbb{R}$

then $u + v = (x_1, 2x) + (y_1, 2y) \underset{\text{under add}}{=} (x_1 + y_1, 2x + 2y) = (x+y, 2(x+y))$ <— which belongs to V

Multp by scalars: If $v = (x, 2x) \in V$, and $k \in \mathbb{R}$

$kv = (kx, k(2x)) = (kx, 2(kx)) \in V$ ✓

Existence: $\boxed{\text{Zero}}$ $(0,0) = (0, 2\cdot 0) \in V$

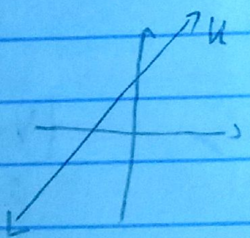
$\boxed{\text{negative}}$ $(x, 2x) + (-x, 2(-x)) = (0,0) = (0, 2\cdot 0)$ ✓

Arithmetic: since we are using the vector operation inherited from $\mathbb{R}$ (where all the arithmetic axioms hold, the arithmetic scalars axioms are ok

So $V = \{(x, 2x) \mid x \in \mathbb{R}\}$ (with operations exactly as in $\mathbb{R}^2$)

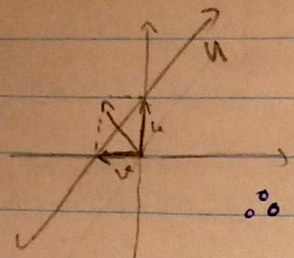
$\underline{\underline{\text{is}}}$ a vector space

Example: $U = \{(x, x+2) \mid x \in \mathbb{R}$, operations are the standard ones



Arithmetic axioms ✓

Closure: (NO!) for example

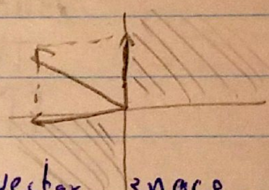


$(0,2) \in U$ and $(-2,0) \in U$ but their sum $(0,2)+(-2,0)$
$= (-2,2) \notin U$

$\therefore$ U is **not** closed under addition

So: U, with standard operation, is not a vector space.

e.g.: $W = \{(x,y) \mid xy \geq 0\}$, usual operation



W is not closed under addition $\therefore$ W is not a vector space

e.g.: $M_{2,2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a,b,c,d \in \mathbb{R} \right\}$

Addn: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$

Multpn: If $k \in \mathbb{R}$, $k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

Closure $+$ ✓
k ✓

Existence 0 ✓ $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M_{2,2}(\mathbb{R})$ works as zero

$-ve$ $\to$ $-\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \in M_{2,2}(\mathbb{R})$

Arithmetic: ✓ ✓ ✓ ✓ ✓

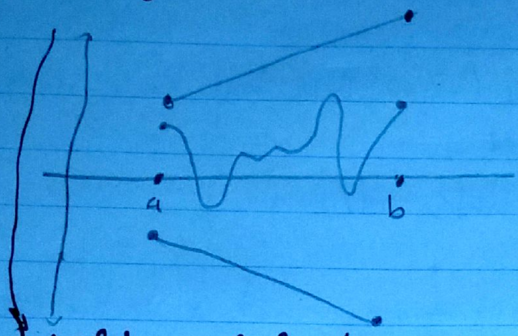
So $M_{2,2}(\mathbb{R})$, with these operation is a vector space.

e.g. Spaces of functions

- If $a,b \in \mathbb{R}$, $[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

$F[a,b] = \{f \mid f : [a,b] \longrightarrow \mathbb{R}$, f is a function from whose domain is $[a,b]$ to $\mathbb{R}$, i.e real valued functions defined on $[a,b]$

$F[a,b]$: The collection of all real valued function defined on $[a,b]$



Addition of functions: Let $f, g \in F[a,b]$.
Their sum is $f+g$, and is defined by
If $x \in [a,b]$, $(f+g)(x) = f(x) + g(x)$

Multp by scalars: If $f \in F[a,b]$ and $k \in \mathbb{R}$, for any $x \in [a,b]$
define $(kf)(x) = k \cdot f(x)$

C $+$ ✓
k ✓

Existence: 0 - Define $z : [a,b] \to \mathbb{R}$ by $z(x) = 0$, for all $x \in [a,b]$
then for all $f \in F[a,b] \to f + z = f$
-ve: If $f \in F[a,b]$ and we define $(-f)(x) = -f(x)$ for $x \in [a,b]$
then $[f + (-f)](x) = f(x) - f(x) = 0$
for all $x \in [a,b]$ i.e $f + (-f) = Z$

Arithmetic: ✓ ✓ ✓ ✓ ✓ ✓

So $F[a,b]$ with these operation is a vector space