1341C.W15 Test 2 version 1.

1. Let $U = \{(x, y, z, w) \in \mathbf{R}^4 \mid x\,y\,w = 0\}$. Then,

A. $(0,0,0,0) \in U$ but U is not closed under multiplication by scalars ✗ ✗
B. U is closed under addition and U is closed under multiplication by scalars ✗✗✗
C. U is closed under addition but U is not closed under multiplication by scalars ✗✗✗
D. U is not closed under addition but U is closed under multiplication by scalars ✓
E. $(0,0,0,0) \notin U$ but U is closed under addition ✗
F. None of the other statements is true.

Clearly, $(0,0,0,0) \in U$

U is closed under mult^n by scalars: if $u=(x,y,z,w) \in U$, and $k \in \mathbb{R}$, then (kx, ky, kz, kw) satisfies $(kx)(ky)(kw)$
$= k^3\, xyw$
$= k^3 \cdot 0$ (since $u \in U$)
$= 0$. (so ✗✗)

However, $(1,0,0,0) \in U$ and $(0,1,0,1) \in U$ but their sum, $(1,1,0,1) \notin U$, so U is not closed under addition. (so ✗✗✗) Hence D is correct

2. Which of the following are subspaces of $\mathbf{R}^3$?

$U = \{(x,\ y,\ z) \mid 2x - y + 3z = 0\}$ is a plane through 0, so is a S.S.
$V = \{(x,\ y,\ z) \mid xy = 0\}$
$W = \{(x,\ y,\ z) \mid 2x = 5z\}$ is a plane through 0, so is a S.S.
$X = \{(x,\ y,\ z) \mid x = y + 3 = 7z\}$ is a line not through $(0,0,0)$, so is not.

A. U and V
B. U, W and X
C. W and X
D. U and W
E. V and X
F. V and W

As "U, V & W" is not an option, the answer must be U & W

Indeed, V is not closed under addition: $(1,0,0)$ and $(0,1,0) \in V$ but their sum, $(1,1,0)$ does not belong to V.

3. Which two of the following are subspaces of $\mathbf{F}[\mathbf{R}] = \{f \mid f\colon \mathbf{R} \to \mathbf{R}\}$? (x)

$S = \{f \in \mathbf{F}[\mathbf{R}] \mid f(0) = 1\}$ This does not contain the zero f^n. ∴ is not a s.s.

$T = \{f \in \mathbf{F}[\mathbf{R}] \mid f(1) = 0\}$ This is a s.s. (we did a similar example in class)

$U = \{f \in \mathbf{F}[\mathbf{R}] \mid f(0)f(1) = 0\}$ This is not a s.s — see example below (xx)

$V = \{f \in \mathbf{F}[\mathbf{R}] \mid f(x) = f(-x), \quad \forall x \in \mathbf{R}\}$ This is. is s.s.

A. S and V. x
B. T and U. xxx
C. S and T. x
D. T and V.
E. S and U. x
F. V and U. xx

Note: if $f(x) = x \quad \forall x \in \mathbb{R}$, then $f \in U$

if $g(x) = x-1, \forall x \in \mathbb{R}$, then $g \in U$

but $(f+g)(x) = 2x-1$ satisfies

$(f+g)(0).(f+g)(1) = -1 \cdot 1 = -1 \neq 0$.

Hence $f+g \notin U$. So U is not closed under addition. (xxx)

V is indeed a subspace of $F[R]$:

1) $O(x) = 0 = O(-x), \ \forall x \in \mathbb{R}$ so $O \in V$

2) If f and g belong to V, then

$(f+g)(x) = f(x)+g(x) = f(-x)+g(-x) = (f+g)(-x), \ \forall x \in \mathbb{R}$,

so $f+g \in V$

3) If $f \in V$ and $k \in \mathbb{R}$, then

$(kf)(x) = k.f(x) = k.f(-x) = (kf)(-x), \forall x \in \mathbb{R}$

So $kf \in V$.

4. Let $W = \{(x, y, z) \in \mathbf{R}^3 \mid x - y + 3z = 0\}$.

a) Explain very briefly why W is a subspace of $\mathbf{R}^3$. (*You will not need to use the Subspace Test - use work we did in class.*)

b) Find a spanning set for W.

c) Give a complete geometric description of W.

(Remember that you must justify your answers.)

a) W is a plane through the origin in $\mathbb{R}^3$ and therefore is a Subspace of $\mathbb{R}^3$

b) $W = \{(y-3z, y, z) \mid y, z \in \mathbb{R}\}$ $(x = y - 3z \text{ iff } (x,y,z) \in W)$

$= \{y(1,1,0) + z(-3,0,1) \mid y, z \in \mathbb{R}\}$

$= \text{span}\{(1,1,0), (-3,0,1)\}$

Hence $\{(1,1,0), (-3,0,1)\}$ is a spanning set for W.

c) W is the plane through 0 with normal $(1,-1,3)$.

5. Let $\mathbf{M}_{2\,2}$ denote the vector space of 2 by 2 matrices with real entries, and define

$$U = \left\{ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \in \mathbf{M}_{2\,2} \mid a, b \in \mathbf{R} \right\}.$$

a) Either check that U is closed under addition, or express U in another form so you can simply state a theorem that guarantees that U is a subspace.

(For (b) and (c) you may assume that U is a subspace of $\mathbf{M}_{2\,2}$.)

b) Find a spanning set for U.

c) Give a matrix $A \in \mathbf{M}_{2\,2}$ such that $A \notin U$.

(Remember that you must justify your answers.)

a) Note that $U = \left\{ a \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$

$= \text{Span} \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ and hence

U is a subspace of M_{22}.

b) From (a), we see that $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is a spanning set for U.

c) The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \notin U$.

6. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you <u>must give an explicit example - with numbers, or functic</u>
- If you say the statement is always true, you must give a clear explanation.

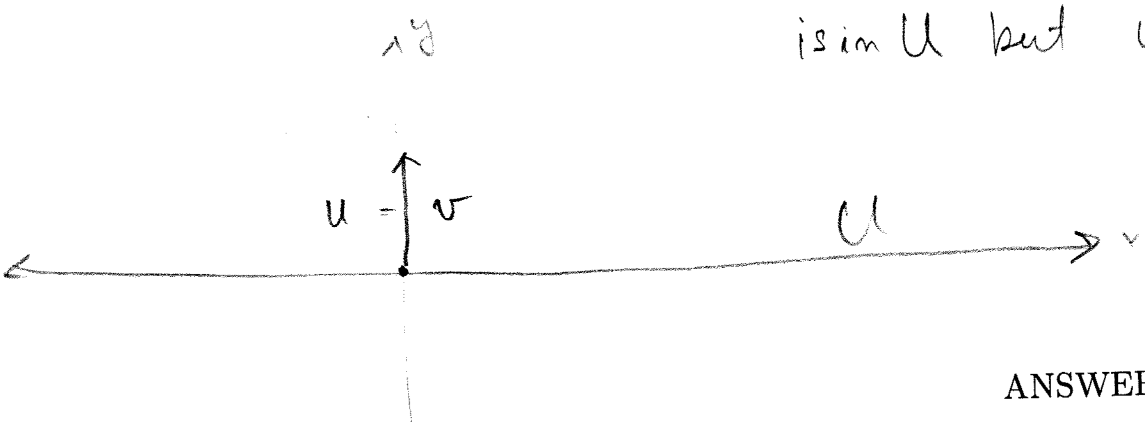
a) $X = \{f \in \mathbf{F}(\mathbf{R}) \mid f(x) \leq 0 \quad \text{for all } x \in \mathbf{R}\}$ is a subspace of $\mathbf{F}(\mathbf{R})$

This is false: If $f(x) = -1 \ \forall x \in \mathbb{R}$, then $f \in X$,
but $(-f)(x) = 1, \forall x \in \mathbb{R}$, so $-f \notin X$. Hence X
is not closed under mult^{n} by scalars.

ANSWER $\boxed{\text{FALSE}}$

b) If u and v are vectors in $\mathbf{R}^2$ and U is a subspace of $\mathbf{R}^2$ with $u - v \in U$, then both u and v belong to U.

Let $U = \{(x,0) \mid x \in \mathbb{R}\}$, $u = (0,1) = v$.
Then neither u nor v is in U but $u - v = (0,0) \in U$



ANSWER $\boxed{\text{FALSE}}$

6 (cont.).

$Y=$

c) $\left\{ \begin{bmatrix} a & a \\ b & c \end{bmatrix} \in \mathbf{M}_{2\,2} \mid a, b, c \in \mathbf{R} \right\}$ is a subspace of $\mathbf{M}_{2\,2}$.

$Y = \text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, and hence

Y is a s.s. of M_{22}

ANSWER TRUE

d) If u, v belong to a vector space V then $\text{span}\{u, v\} = \text{span}\{u, u + v\}$.

Let $w \in \text{span}\{u, v\}$ Then $w = au + bv$ for some $a, b \in \mathbb{R}$

We then (after some calcn*) see that $w = (a-b)u + b(u+v)$
$\in \text{span}\{u, u+v\}$

Hence $\text{span}\{u,v\} \subseteq \text{span}\{u, u+v\}$.

Now Suppose $w = cu + d(u+v)$ for some $c, d \in \mathbb{R}$.
Then $w = (c+d)u + dv \in \text{span}\{u, v\}$. Hence

$\text{span}\{u, u+v\} \subseteq \text{span}\{u, v\}$
Thus $\text{span}\{u,v\} = \text{span}\{u, u+v\}$

ANSWER TRUE

* $au + bv = cu + d(u+v) \Leftarrow au + bv = (c+d)u + dv \Leftarrow c+d = a$ &
$d = b \iff d = b$ and $c = a - d = a - b$

7. [Bonus] Give the set $U = \{(x-2, x) \mid x \in \mathbf{R}\}$ the *non-standard* operations

$$(x,\ y) \oplus (x',\ y') = (x + x' + 2,\ y + y') \quad \text{(vector addition)}$$

and

$$k \odot (x,\ y) = (kx + 2k - 2,\ ky) \quad \text{(multiplication by scalars).}$$

a) Prove that U is closed under the operation of vector addition defined above.

b) Show that U *has* a zero vector. (i.e. find it and show it works.).

a) Let $u_1 = (x-2, x)$ and $u_2 = (x'-2, x')$ lie in U.

Then $u_1 \oplus u_2 = ((x-2) + (x'-2) + 2,\ x + x')$

$= ((x+x') - 2,\ x + x') \in U.$

Hence U is closed under add^n.

b) We claim $\hat{0} = (-2, 0)$ is a zero vector:

Let $u = (x-2, x) \in U$. Then

$u \oplus \hat{0} = (x-2, x) \oplus (-2, 0)$

$= ((x-2) + (-2) + 2,\ x + 0)$

$= (x-2, x)$

$= u.$

Hence $\hat{0}$ is a zero vector.