1. Let $X = \{(a, b, c) \in \mathbf{R}^3 \mid bc = 0\}$. Then,   Note: $(0,0,0) \in X$ (so D is false)

A. X is closed under addition and X is closed under multiplication by scalars ✗

B. X is closed under addition but X is not closed under multiplication by scalars ✗

C. X is not closed under addition but X is closed under multiplication by scalars ✓

D. $(0, 0, 0) \notin X$ but X is closed under addition ✗

E. $(0, 0, 0) \in X$ but X is not closed under multiplication by scalars ✗

F. None of the other statements is true.

- Let $u = (0,1,0)$ and $v = (0,0,1)$. Then $u, v \in X$ but $u+v = (0,1,1) \notin X$. Hence X is not closed under addn.
- If $u = (a,b,c) \in X$ and $k \in \mathbb{R}$, then $ku = (ka, kb, kc)$ and $bc = 0$ $\Rightarrow (kb)(kc) = k^2 bc = 0$ $\Rightarrow ku \in X$. Hence X is closed under multn by scalars.

2. Which of the following are subspaces of $\mathbf{R}^3$?

(1)=$\{(x,\ x+y,\ x+2y) \in \mathbf{R}^3 \mid x, y \in \mathbf{R}\}$ ✓
(2)=$\{(x,\ y,\ z) \in \mathbf{R}^3 \mid x-2 = y-3 = z\}$ ✗
(3)=$\{(x,\ y,\ z) \in \mathbf{R}^3 \mid xyz = 0\}$ ✗
(4)=$\{(x,\ y,\ z) \in \mathbf{R}^3 \mid x-y-z = 0\}$ ✓

A. (1) and (2)
B. (1), (3) and (4)
C. (3) and (4)
D. (1) and (3)
E. (1) and (4)
F. (2) and (3)

(1) $= \operatorname{span}\{(1,1,1), (0,1,2)\}$ and is therefore a s.s. of $\mathbb{R}^3$

(2) does not contain $(0,0,0)$ (and so is not a s.s. of $\mathbb{R}^3$)

(3) contains $(1,0,0)$ and $(0,1,1)$ but not their sum, $(1,1,1)$. Hence (3) is not a s.s. of $\mathbb{R}^3$

(4) is a plane through $(0,0,0)$ and hence is a s.s. of $\mathbb{R}^3$

3. Which of the following are subspaces of $\mathbf{M}_{2\,2}(\mathbf{R})$?

X A=$\left\{ \begin{bmatrix} a & 1 \\ b & b \end{bmatrix} \in \mathbf{M}_{2\,2} \mid a, b \in \mathbf{R} \right\}$ X $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin A$

✓ B $\left\{ \begin{bmatrix} a & b \\ 2a & c \end{bmatrix} \in \mathbf{M}_{2\,2} \mid a, b, c \in \mathbf{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ ✓S.S

X C. $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{2\,2} \mid ab = 1;\ \overset{c}{\cancel{a}}, d \in \mathbf{R} \right\}$

This set does contain the zero matrix, so it not a subspace.

X D. $\left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in \mathbf{M}_{2\,2} \mid a, b, c \text{ integers.} \right\}$ — contains $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ but not $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

✓ E. $\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{2\,2} \mid cd = 0;\ a, b \in \mathbf{R} \right\}$ — contains $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ but not their sum

F. None of the above.

4. Let $W = \{(x, y, z) \in \mathbf{R}^3 \mid x + 5y - 4z = 0\}$.

a) Explain very briefly why W is a subspace of $\mathbf{R}^3$. (*You will not need to use the Subspace Test - use work we did in class.*)

b) Find a spanning set for W.

c) Give a complete geometric description of W.

(Remember that you must justify your answers.)

a) W is a plane through O and hence is a subspace of $\mathbb{R}^3$

b) Note that $W = \{(-5y + 4z, y, z) \mid y, z \in \mathbb{R}\}$
$= \text{span}\{(-5, 1, 0), (4, 0, 1)\}$,

and hence $\{(-5, 1, 0), (4, 0, 1)\}$ spans W.

c) W is the plane through O with normal $(1, 5, -4)$.

5. Let $\mathbf{M}_{22}$ denote the vector space of 2 by 2 matrices with real entries, and define

$$U = \left\{ \begin{bmatrix} 0 & a \\ b & 2a \end{bmatrix} \in \mathbf{M}_{22} \mid a, b \in \mathbf{R} \right\}.$$

$$\left(\begin{bmatrix} 0 & a \\ b & 2a \end{bmatrix} = a \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right)$$

a) Either check that U is closed under addition, or express U in another form so you can simply state a theorem that guarantees U is a subspace.

(For (b) and (c) you may assume that U is a subspace of $\mathbf{M}_{22}$.)

b) Find a spanning set for U.

c) Give a matrix $A \in \mathbf{M}_{22}$ such that $A \notin U$.

(Remember that you must justify your answers.)

a) Note that $U = \text{span}\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

and hence is a s.s of M_{22}.

b) From (a) we see that $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

spans U.

c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \notin U$.

6. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you <u>must give an explicit example - with numbers!</u>
- If you say the statement is always true, you must give a clear explanation.

a) $X = \{f \in \mathbf{F}(\mathbf{R}) \mid f(2) \leq 0\}$ is a subspace of $\mathbf{F}(\mathbf{R})$

If we define $f(x) = -1$, $\forall x \in \mathbb{R}$, then $f \in X$

but $(-f)(x) = 1$, $\forall x \in \mathbb{R}$ shows that $-f \notin X$.

Hence X is not a s.s. of $F(\mathbb{R})$

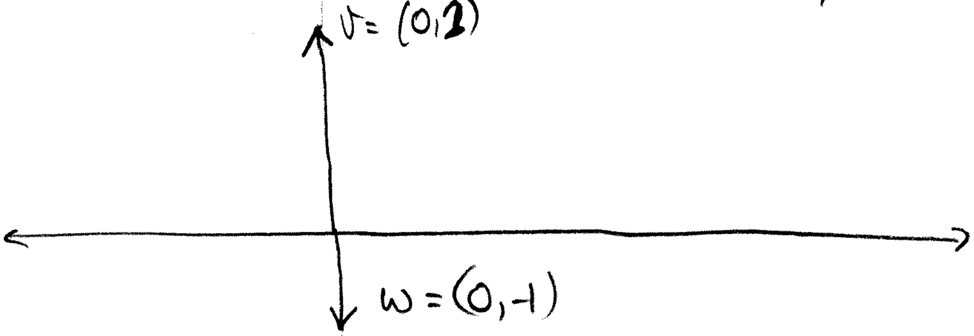
ANSWER $\boxed{\text{FALSE}}$

b) If v and w are vectors in $\mathbf{R}^2$ and X is a subspace of $\mathbf{R}^3$ with $v + 2w \in X$, then both v and w belong to X.

Let $X = \{(x, 0) \mid x \in \mathbb{R}\}$, $v = (0,2)$, $w = (0,-1)$

then $v + 2w = (0,0) \in X$

but neither v nor w belongs to X.



ANSWER $\boxed{\text{FALSE}}$

6 (cont.).

c) $W = \left\{ \begin{bmatrix} a & b \\ a & c \end{bmatrix} \in \mathbf{M}_{22} \mid a, b, c \in \mathbf{R} \right\}$ is a subspace of $\mathbf{M}_{22}$.

Note $W = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

and hence W is a subspace of M_{22}

ANSWER $\boxed{\text{TRUE}}$

d) If v and w belong to a vector space V then $\text{span}\{v, w\} = \text{span}\{v + w, v\}$.

If $u \in \text{span}\{v, w\}$, then $u = av + bw$ (for $a, b \in \mathbb{R}$)

$= b(v+w) + (a-b)v$

$\in \text{span}\{v+w, w\}$

Hence $\text{span}\{v, w\} \subseteq \text{span}\{v+w, v\}$.

If $u \in \text{span}\{v+w, v\}$, then $u = a(v+w) + bv$ for $a, b \in \mathbb{R}$

So $u = (a+b)v + aw \in \text{span}\{v, w\}$

Hence $\text{span}\{v+w, v\} \subseteq \text{span}\{v, w\}$.

ANSWER $\boxed{\text{TRUE}}$

Hence $\text{span}\{v, w\} = \text{span}\{v+w, v\}$

$(x-4, x)$

7. [Bonus] Give the set $U = \{(x - 4, x) \mid x \in \mathbf{R}\}$ the *non-standard* operations

$$(x,\ y) \oplus (x',\ y') = (x + x' + 4,\ y + y') \quad \text{(vector addition)}$$

$\hat{0} = (-4, 0)$

and

$$k \odot (x,\ y) = (kx + 4k - 4,\ ky) \quad \text{(multiplication by scalars).}$$

a) Prove that U is closed under the operation of vector addition defined above.

b) Show that U *has* a zero vector. (i.e. find it and show it works.)

a) If $u = (x-4, x)$ and $v = (x'-4, x') \in U$ then

$$u \oplus v = ((x-4) + (x'-4) + 4,\ x + x')$$
$$= ((x+x') - 4,\ x + x') \in U.$$

b) We claim $\hat{0} = (-4, 0)$: Let $u = (x-4, x) \in U$.

Then $$u \oplus \hat{0} = (x - 4 + (-4) + 4,\ 0 + x)$$
$$= (x-4, x)$$
$$= u.$$

Hence $\hat{0}$ is a zero for U.