PART A (35 marks)

A1. If $\log_3 x - \log_3 x^2 = 2$, find x.

A: 6	B: 3	C: $\frac{1}{3}$	D: 9	E: $\frac{1}{9}$

Solution:

$$\log_3 x - \log_3 x^2 = 2 \quad\Rightarrow\quad \log_3\left(\frac{x}{x^2}\right) = 2 \quad\Rightarrow\quad \log_3\left(\frac{1}{x}\right) = 2 \quad\Rightarrow\quad \frac{1}{x} = 3^2 \quad\Rightarrow\quad x = \frac{1}{3^2} = \frac{1}{9}$$

A2. If $f(x) = 4e^{\sqrt{x}}$, find $f'(4)$.

A: $\frac{1}{2}e^2$	B: e^2	C: $2e^2$	D: $4e^2$	E: $8e^2$

Solution: Using the chain rule (and the power rule if you don't remember the derivative of $\sqrt{x}$) we get

$$f'(x) = \frac{d}{dx}\left[4e^{\sqrt{x}}\right] = 4e^{\sqrt{x}}\left[\frac{d}{dx}\left(\sqrt{x}\right)\right] = 4e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right) = \frac{4e^{\sqrt{x}}}{2\sqrt{x}}$$

so we have $f'(4) = \dfrac{4e^{\sqrt{4}}}{2\sqrt{4}} = \dfrac{4e^2}{2(2)} = e^2$.

A3. If $f(x) = 2^{-x}$, find $f'(0)$.

A: 0	B: 1	C: -1	D: $-\frac{1}{\ln 2}$	E: $-\ln 2$

Solution: We know that the derivative of b^x is $b^x \ln b$, so (using the chain rule) we get

$$f'(x) = \frac{d}{dx}\left[2^{-x}\right] = 2^{-x}\ln 2\left[\frac{d}{dx}(-x)\right] = -2^{-x}\ln 2$$

Therefore $f'(0) = -2^0 \ln 2 = -\ln 2$.

A4. Find the slope of the tangent line to the graph of $y = e^{2x}$ at the point where $x = \ln 3$.

A: 8	B: 6	C: 12	D: 18	E: 9

Solution: Of course the slope of the tangent line to $y = f(x)$ at the point where $x = a$ is $f'(a)$. For $f(x) = e^{2x}$ we have (using the chain rule) $f'(x) = 2e^{2x}$ and so the slope of the tangent line at the point where $x = \ln 3$ is

$$f'(\ln 3) = 2e^{2\ln 3} = 2e^{\ln 3^2} = 2(3^2) = 18$$

A5. If $\dfrac{8x-11}{(x-2)(x-1)} = \dfrac{A}{x-2} + \dfrac{B}{x-1}$, find B.

A: 1	B: 2	C: 3	D: 4	E: 5

Solution: We have $\dfrac{A}{x-2} + \dfrac{B}{x-1} = \dfrac{A(x-1)+B(x-2)}{(x-2)(x-1)}$ so we need $A(x-1)+B(x-2) = 8x-11$. To find B we use $x = 1$, so that the multiplier on A is 0. This gives

$$A(1-1) + B(1-2) = 8(1) - 11 \quad\Rightarrow\quad 0A + (-1)B = 8 - 11 \quad\Rightarrow\quad -B = -3 \quad\Rightarrow\quad B = 3$$

A6. Find $\int 12x^3 \ln x\, dx$.

A: $x^4\left(3\ln x - \frac{1}{4}\right) + C$	B: $3x^3(x\ln x - 1) + C$	C: $12x^4(\ln x - 1) + C$
D: $3x^4\left(\ln x - \frac{1}{4}\right) + C$	E: $\frac{3x^4(\ln x)^2}{2} + C$	

Solution: We need integration by parts. We let $u = \ln x$ and $dv = 12x^3dx$, so that $du = \left(\frac{1}{x}\right) dx$ and $v = 3x^4$. (That is, $dv = 12x^3dx = 3(4x^3)dx$ and x^4 is an antiderivative of $4x^3$.) This gives

$$\begin{aligned}\int 12x^3 \ln x\, dx &= \int u\, dv = uv - \int v\, du = 3x^4 \ln x - \int 3x^4 \left(\frac{1}{x}\right) dx \\ &= 3x^4 \ln x - \int 3x^3\, dx = 3x^4 \ln x - 3\left(\frac{x^4}{4}\right) + C = 3x^4\left(\ln x - \frac{1}{4}\right) + C\end{aligned}$$

A7. Evaluate $\int_e^{e^3} \frac{1}{x\ln x}\, dx$.

A: $\ln 3$	B: $e^3 - e$	C: $\frac{1}{3}$	D: $-\frac{2}{3}$	E: 0

Solution: We need the substitution rule. Let $u = \ln x$, so that $du = \frac{1}{x}dx$. Then when $x = e$, $u = \ln e = 1$ and when $x = e^3$, $u = \ln e^3 = 3$. Therefore we get

$$\int_e^{e^3} \frac{1}{x\ln x}\, dx = \int_e^{e^3} \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) dx = \int_1^3 \frac{1}{u}\, du = [\ln u]_1^3 = \ln 3 - \ln 1 = \ln 3 - 0 = \ln 3$$

A8. Evaluate $\int_1^{\infty} \frac{-1}{x^5}\, dx$.

A: $\frac{1}{4}$	B: $-\frac{1}{4}$	C: 1	D: -1	E: diverges

Solution:

$$\int_1^{\infty} \frac{-1}{x^5}\, dx = \lim_{b\to\infty} \int_1^b (-x^{-5})\, dx = \lim_{b\to\infty} \left[\frac{-x^{-4}}{-4}\right]_1^b = \lim_{b\to\infty} \left[\frac{1}{4x^4}\right]_1^b = \left[\lim_{b\to\infty}\left(\frac{1}{4b^4}\right)\right] - \frac{1}{4(1)^4} = 0 - \frac{1}{4} = -\frac{1}{4}$$

(since as $b \to \infty$, $b^4 \to \infty$, so $4b^4 \to \infty$ and $\frac{1}{4b^4} \to 0$.)

A9. Determine which one of the following integrals represents the area of the region bounded by $y = x^2$ and $y = 4$.

A: $\int_0^2 (4 - x^2)\, dx$	B: $\int_0^4 (4 - x^2)\, dx$	C: $\int_0^2 \sqrt{y}\, dy$	D: $\int_0^4 \sqrt{y}\, dy$	E: $\int_0^4 2\sqrt{y}\, dy$

Solution: The functions $y = x^2$ and $y = 4$ intersect when $x^2 = 4$, at $x = -2$ and $x = 2$, and between these x-values, $y = 4$ lies above $y = x^2$. (For instance, at $x = 0$, $x^2 = 0 < 4$.) Therefore the region bounded by

$y = x^2$ and $y = 4$ has $y = 4$ as upper boundary and $y = x^2$ as lower boundary, and goes from $x = -2$ on the left to $x = 2$ on the right. Thus the area of the region is given by

$$\text{Area} = \int_{\text{left}}^{\text{right}} (\text{upper} - \text{lower})\, dx = \int_{-2}^{2} (4 - x^2)\, dx$$

Unfortunately, this is not one of the answer choices, which means the correct answer choice does not express the area using vertical slicing, so it must be using horizontal slicing. When horizontal slicing is used, we consider the region as being bounded on the left by $x = -\sqrt{y}$ and on the right by $x = \sqrt{y}$ (i.e. the left and right halves of the parabola $x = \pm\sqrt{y}$ or $x^2 = y$), and running from $y = 0$ at the bottom to $y = 4$ at the top. This gives the area as

$$\text{Area} = \int_{\text{bottom}}^{\text{top}} (\text{rightmost} - \text{leftmost})\, dy = \int_0^4 [\sqrt{y} - (-\sqrt{y})]\, dy = \int_0^4 2\sqrt{y}\, dy$$

A10. Find the area of the region bounded by $y = x^2 - x$ and $y = x$.

A: $\frac{8}{3}$	B: $-\frac{8}{3}$	C: $\frac{4}{3}$	D: $-\frac{4}{3}$	E: $\frac{16}{3}$

Solution: The functions $y = x^2 - x$ and $y = x$ intersect when $x^2 - x = x$, i.e. when $x^2 - 2x = 0$, so when $x(x-2) = 0$, which is when $x = 0$ and when $x = 2$. On the interval $[0, 2]$, $x^2 - x < x$ since for instance at $x = 1$ we have $x^2 - x = 1^2 - 1 = 0 < 1 = x$. Therefore $y = x$ is the upper boundary of the region, and $y = x^2 - x$ is the lower boundary, so the area is given by

$$\begin{aligned}\text{Area} &= \int_{\text{left}}^{\text{right}} (\text{upper} - \text{lower})\, dx = \int_0^2 [x - (x^2 - x)]\, dx = \int_0^2 (2x - x^2)\, dx \\ &= \left[x^2 - \frac{x^3}{3}\right]_0^2 = \left(2^2 - \frac{2^3}{3}\right) - \left(0^2 - \frac{0^3}{3}\right) = 4 - \frac{8}{3} - 0 = \frac{12}{3} - \frac{8}{3} = \frac{4}{3}\end{aligned}$$

A11. Find the volume of the solid of revolution obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 2$ and $x = 0$ about the x-axis.

A: 6π	B: 8π	C: $\frac{8\pi}{3}$	D: $\frac{16\pi}{3}$	E: $\frac{32\pi}{5}$

Solution: The line $x = 0$ is the y-axis, and the curve $y = \sqrt{x}$ lies entirely to the right of the y-axis, so $x = 0$ is the left edge of the region. The curve $y = \sqrt{x}$ intersects the line $y = 2$ when $\sqrt{x} = 2$, i.e. when $x = 4$, so this is the rightmost extent of the region. Between $x = 0$ and $x = 4$, the curve $y = \sqrt{x}$ lies below the line $y = 2$ (for instance at $x = 1$ we have $\sqrt{x} = 1 < 2$), so $y = 2$ is the upper boundary of the region and $y = \sqrt{x}$ is the lower boundary. Since the region is being revolved about the x-axis (a horizontal axis of revolution), we need vertical slicing, and since the x-axis is not a boundary of the region, we need the method of washers. A vertical slice of the region has upper edge $y = 2$ and lower edge $y = \sqrt{x}$, so the washer formed by revolving this slice about the x-axis has outer radius $R = 2$ and inner radius $r = \sqrt{x}$. Since the region runs from $x = 0$ to $x = 4$, the volume of the solid obtained by revolving the entire region about the x-axis is

$$\begin{aligned}\text{Volume} &= \pi \int_0^4 (R^2 - r^2)\, dx = \pi \int_0^4 [(2)^2 - (\sqrt{x})^2]\, dx = \pi \int_0^4 (4 - x)\, dx \\ &= \pi \left[4x - \frac{x^2}{2}\right]_0^4 = \pi \left[\left(4(4) - \frac{4^2}{2}\right) - \left(4(0) - \frac{0^2}{2}\right)\right] = \pi(16 - 8 - 0) = 8\pi\end{aligned}$$

A12. Find the volume of the solid of revolution obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 2$ and $x = 0$ about the y-axis.

A: 6π	B: 8π	C: $\frac{8\pi}{3}$	D: $\frac{16\pi}{3}$	E: $\frac{32\pi}{5}$

Solution: This time we have the same region as in the previous question, but it is being revolved about the y-axis, a vertical axis of revolution, so we need horizontal slicing. And since the axis of revolution is a boundary of the region, we use the method of disks. To use horizontal slicing, we need to express the left and right boundaries of the region in $x = f(y)$ form. The left boundary is $x = 0$, which is already in the correct form. The right boundary is the curve $y = \sqrt{x}$, which is the upper half of the parabola $x = y^2$. The bottom of the region is at $y = 0$ (where $y = \sqrt{x}$ intersects $x = 0$) and the top of the region is at $y = 2$ (the line forming the upper boundary of the region). So the region lies between $x = 0$ on the left and $x = y^2$ on the right, from $y = 0$ at the bottom to $y = 2$ at the top. A horizontal slice of the region has left edge $x = 0$ and right edge $x = y^2$, so the disk formed by revolving this slice about the y-axis has radius $r = y^2$. Therefore the volume of the solid formed by revolving the entire region about the y-axis is given by

$$\text{Volume } = \pi \int_0^2 r^2 dy = \pi \int_0^2 (y^2)^2 dy = \pi \int_0^2 y^4 dy = \pi \left[\frac{y^5}{5} \right]_0^2 = \pi \left[\frac{2^5}{5} - \frac{0^5}{5} \right] = \frac{32\pi}{5}$$

A13. If $f(x, y) = x^2 - 3xy + y^3$, find $f_y(3, 1)$.

A: -7	B: 3	C: 24	D: -6	E: 1

Solution: To find $f_y(x, y)$, we differentiate $f(x, y)$ with respect to y, treating x as a constant. We get

$$f_y(x, y) = \frac{\partial}{\partial y}(x^2 - 3xy + y^3) = 0 - 3x(1) + 3y^2 = 3y^2 - 3x = 3(y^2 - x)$$

This gives $f_y(3, 1) = 3(1^2 - 3) = 3(-2) = -6$.

A14. If $f(x, y) = y^x$, find $f_{yx}(x, y)$.

A: $y^x \ln y$	B: xy^{x-1}	C: y^{x-1}	D: $y^{x-1}(1 + x \ln y)$	E: y^x

Solution: To find $f_{yx}(x, y)$, which is the partial derivative with respect to x of $f_y(x, y)$, we first differentiate with respect to y, and then with respect to x. For $f_y(x, y)$, we have the form y raised to a constant power (since x is treated as a constant), so we use the power rule. That is,

$$f_y(x, y) = \frac{\partial}{\partial y}(y^x) = xy^{x-1}$$

And now, to find the partial of this function with respect to x, we have a product of two terms which both contain x, so we need the product rule. And since we treat y as a constant when finding the partial derivative with respect to x, the second term in the product has the form "constant raised to the power x", so we use the fact that the derivative of b^x is $b^x \ln b$ (with the chain rule, since the exponent is not just x). We get

$$\begin{aligned} f_{yx}(x, y) &= \frac{\partial}{\partial x}(xy^{x-1}) = \left[\frac{\partial}{\partial x}(x) \right] (y^{x-1}) + x \left[\frac{\partial}{\partial x}(y^{x-1}) \right] = 1(y^{x-1}) + x \left[y^{x-1}(\ln y) \left(\frac{\partial}{\partial x}(x - 1) \right) \right] \\ &= y^{x-1} + y^{x-1}[x(\ln y)(1)] = y^{x-1}(1 + x \ln y) \end{aligned}$$

A15. Let $f(x, y) = ye^x - 3x - y$. Find the only critical point of $f(x, y)$.

A: $(0, 0)$	B: $(0, 3)$	C: $(1, 1)$	D: $(3, 1)$	E: $(-3, -1)$

Solution: Critical points of a function of two variables are those points on the function at which both first partials are 0. So first, we need to find the first partials of $f(x, y)$. We get

$$f_x(x, y) = \frac{\partial}{\partial x}(ye^x - 3x - y) = y(e^x) - 3(1) - 0 = ye^x - 3$$

$$\text{and } f_y(x, y) = \frac{\partial}{\partial y}(ye^x - 3x - y) = e^x(1) - 0 - 1 = e^x - 1$$

To have $f_y(x, y) = 0$ we need $e^x - 1 = 0$, i.e. $e^x = 1$ so we must have $x = 0$. (That is, we need $\ln e^x = \ln 1$ so that $x = \ln 1 = 0$.) And to have $f_x(x, y) = 0$ we need $ye^x - 3 = 0$ so that $ye^x = 3$. And since we must have both $f_y(x, y) = 0$ and $f_x(x, y) = 0$ at the same time, we know that $e^x = 1$, which gives $ye^x = y(1) = y$ so we see that $y = 3$. That is, the only time that both first partials of $f(x, y)$ are 0 is when $x = 0$ and $y = 3$, so $(x, y) = (0, 3)$ is the only critical point of the function.

A16. Find all the critical points of the function $f(x, y) = 3x - x^3 - 3xy^2$.

A: $(0, 0)$	B: $(1, 1), (-1, -1)$	C: $(1, 0), (-1, 0), (0, 1), (0, -1)$
D: $(0, 0), (-1, 1), (1, 1)$	E: $(1, -1), (1, 1), (-1, 1), (-1, -1)$	

Solution: The critical points of the function $f(x, y)$ are all those points for which both first partials have the value 0 (at the same time). We start by finding the first partials:

$$f_x(x, y) = \frac{\partial}{\partial x}(3x - x^3 - 3xy^2) = 3 - 3x^2 - 3y^2$$

$$f_y(x, y) = \frac{\partial}{\partial y}(3x - x^3 - 3xy^2) = 0 + 0 - 3x(2y) = -6xy$$

In order to have $-6xy = 0$ we must either have $x = 0$ or have $y = 0$. That is, $f_y(x, y) = 0$ whenever $x = 0$ or $y = 0$, so we need to find the points which make $f_x(x, y) = 0$ when $x = 0$, and also when $y = 0$.
If $x = 0$ then we have $f_x(x, y) = 3 - 3(0^2) - 3y^2 = 3 - 3y^2$, so requiring $f_x(x, y) = 0$ means that we need $3 - 3y^2 = 0$, i.e. $3y^2 = 3$ and so we need $y^2 = 1$. This is true whenever $y = \pm 1$, so this gives two critical points: $x = 0$ and $y = \pm 1$, i.e. $(0, -1)$ and $(0, 1)$.
If $y = 0$ then we have $f_x(x, y) = 3 - 3x^2 - 3(0^2) = 3 - 3x^2$, so requiring $f_x(x, y) = 0$ means that we need $3 - 3x^2 = 0$, i.e. $3x^2 = 3$ and so we need $x^2 = 1$. This is true whenever $x = \pm 1$ and so we get two more critical points: $x = \pm 1$ and $y = 0$, i.e. $(-1, 0)$ and $(1, 0)$.

Use the following information for questions 17, 18 and 19.

$$\begin{aligned} f(x, y) &= x^3 - 6xy - y^2 \\ f_x(x, y) &= 3x^2 - 6y \\ f_y(x, y) &= -6x - 2y \\ f_{xx}(x, y) &= 6x \\ f_{yy}(x, y) &= -2 \\ f_{xy}(x, y) &= -6 \end{aligned}$$

A17. Which one of the following is true for the point $(1, -3)$?

A:	$(1, -3)$ is not a critical point of $f(x, y)$.
B:	$f(x, y)$ has a saddle point at $(1, -3)$.
C:	$f(x, y)$ has a local minimum at $(1, -3)$.
D:	$f(x, y)$ has a local maximum at $(1, -3)$.
E:	The second partials test yields no information.

Solution: For the point $(x, y) = (1, -3)$ we see that

$$f_x(1, -3) = 3(1)^2 - 6(-3) = 3 + 18 = 21$$

Since $f_x(1, -3) \neq 0$, then it is not true that both first partials are 0 at the point $(1, -3)$, so this is not a critical point of $f(x, y)$.

A18. Which one of the following is true for the point $(-6, 18)$?

A:	$(-6, 18)$ is not a critical point of $f(x, y)$.
B:	$f(x, y)$ has a saddle point at $(-6, 18)$.
C:	$f(x, y)$ has a local minimum at $(-6, 18)$.
D:	$f(x, y)$ has a local maximum at $(-6, 18)$.
E:	The second partials test yields no information.

Solution: For the point $(x, y) = (-6, 18)$, we have

$$\begin{aligned} f_x(-6, 18) &= 3(-6)^2 - 6(18) = 3(36) - 6(18) = 108 - 108 = 0 \\ \text{and } f_y(-6, 18) &= -6(-6) - 2(18) = 36 - 36 = 0 \end{aligned}$$

Since $f_x(-6, 18) = f_y(-6, 18) = 0$, we see that $(x, y) = (-6, 18)$ **is** a critical point of $f(x, y)$. We perform the second partials (i.e. second derivative) test. We have

$$D(x, y) = [f_{xx}(x, y)][f_{yy}(x, y)] - [f_{xy}(x, y)]^2 = (6x)(-2) - (-6)^2 = -12x - 36$$

This gives $D(-6, 18) = -12(-6) - 36 = 72 - 36 = 36$. Since $D(-6, 18) > 0$ and $f_{xx}(-6, 18) = 6(-6) = -36 < 0$, we see that $f(x, y)$ has a local maximum at $(x, y) = (-6, 18)$.

A19. Which one of the following is true for the point $(0, 0)$?

A:	$(0, 0)$ is not a critical point of $f(x, y)$.
B:	$f(x, y)$ has a saddle point at $(0, 0)$.
C:	$f(x, y)$ has a local minimum at $(0, 0)$.
D:	$f(x, y)$ has a local maximum at $(0, 0)$.
E:	The second partials test yields no information.

Solution: For the point $(x, y) = (0, 0)$, we have

$$\begin{aligned} f_x(0, 0) &= 3(0)^2 - 6(0) = 0 - 0 = 0 \\ \text{and } f_y(0, 0) &= -6(0) - 2(0) = 0 - 0 = 0 \end{aligned}$$

Since $f_x(0, 0) = f_y(0, 0) = 0$, we see that $(x, y) = (0, 0)$ **is** a critical point of $f(x, y)$. We perform the second partials test. We still have $D(x, y) = -12x - 36$, as in the previous question. This time we get $D(0, 0) = -12(0) - 36 = -36$. Since $D(0, 0) < 0$, we see that $f(x, y)$ has a saddle point at $(x, y) = (0, 0)$.

A20. If the method of Lagrange multipliers is used to maximize the function $f(x, y) = xy + 10$ subject to the constraint $x^2 + 9y^2 = 18$, what system of equations must be solved?

A: $xy + 10 + \lambda(x^2 + 9y^2 - 18) = 0$	B: $2x + \lambda y = 0$, $18y + \lambda x = 0$, $xy + 10 = 0$	C: $y + 2x\lambda = 0$, $x + 18y\lambda = 0$, $x^2 + 9y^2 - 18 = 0$
D: $xy + 2\lambda x = 0$, $xy + 2\lambda y = 0$, $x^2 + 9y^2 - 18 = 0$	E: $xy + 10 = 0$, $x^2 + 9y^2 - 18 = 0$	

Solution: The Lagrange function for this problem is

$$F(x, y, \lambda) = xy + 10 + \lambda(x^2 + 9y^2 - 18)$$

The system which needs to be solved is the system of equations which states that each of the first partials of this function must equal 0. The first partials of $F(x, y, \lambda)$ are

$$\begin{aligned} F_x(x, y, \lambda) &= \frac{\partial}{\partial x}[xy + 10 + \lambda(x^2 + 9y^2 - 18)] = y + 0 + \lambda(2x + 0 - 0) = y + 2x\lambda \\ F_y(x, y, \lambda) &= \frac{\partial}{\partial y}[xy + 10 + \lambda(x^2 + 9y^2 - 18)] = x + 0 + \lambda(0 + 9(2y) - 0) = x + 18y\lambda \\ \text{and } F_\lambda(x, y, \lambda) &= \frac{\partial}{\partial \lambda}[xy + 10 + \lambda(x^2 + 9y^2 - 18)] = 0 + 0 + 1(x^2 + 9y^2 - 18) = x^2 + 9y^2 - 18 \end{aligned}$$

and so the system of equations which must be solved is

$$\begin{aligned} y + 2x\lambda &= 0 \\ x + 18y\lambda &= 0 \\ \text{and } x^2 + 9y^2 - 18 &= 0 \end{aligned}$$

A21. In using Lagrange's method to solve a particular constrained optimization problem, the following system of equations needs to be solved:

$$\begin{aligned} 1 + 2\lambda x &= 0 \\ 3 + 2\lambda y &= 0 \\ 2x^2 + y^2 &= 11 \end{aligned}$$

Find all possible solutions to this system.

A: $(\sqrt{5}, 1)$ and $(-\sqrt{5}, -1)$	B: $(\sqrt{5}, -1)$ and $(-\sqrt{5}, 1)$	C: $(1, -3)$ and $(-1, 3)$
D: $(1, 3)$ and $(-1, -3)$	E: No solutions.	

Solution: Notice neither x nor y can be zero, since these would give $1 + 0 = 0$ or $3 + 0 = 0$ in the first two equations. Therefore we can rearrange the first two equation to $\lambda = \frac{-1}{2x}$ and also $\lambda = \frac{-3}{2y}$. Equating these two expressions for λ gives

$$-\frac{1}{2x} = -\frac{3}{2y} \quad \Rightarrow \quad 2y = 6x \quad \Rightarrow \quad y = 3x$$

Using this in the third equation we have

$$2x^2 + y^2 = 11 \quad \Rightarrow \quad 2x^2 + (3x)^2 = 11 \quad \Rightarrow \quad 2x^2 + 9x^2 = 11 \quad \Rightarrow \quad 11x^2 = 11 \quad \Rightarrow \quad x^2 = 1$$

To satisfy $x^2 = 1$ we can have $x = -1$ or $x = 1$. And of course we need $y = 3x$, so when $x = -1$ we must have $y = -3$ and when $x = 1$ we must have $y = 3$. Therefore the only possible solutions are $(x, y) = (-1, -3)$ and $(x, y) = (1, 3)$.

A22. Find the maximum value of $f(x,y) = xy$ subject to the constraint $x^2 + y^2 = 4$.

A: 2	B: -2	C: 4	D: -4	E: 0

Solution: We use the Lagrange method. The Lagrange function is $F(x,y,\lambda) = xy + \lambda(x^2 + y^2 - 4)$. The first partials of this function are

$$\begin{aligned} F_x(x,y,\lambda) &= y(1) + \lambda(2x + 0 - 0) = y + 2\lambda x \\ F_y(x,y,\lambda) &= x(1) + \lambda(0 + 2y - 0) = x + 2\lambda y \\ F_\lambda(x,y,\lambda) &= x^2 + y^2 - 4 \end{aligned}$$

Therefore the candidates for points at which the maximum value of $f(x,y) = xy$ might be attained, while satisfying the constraint $x^2 + y^2 = 4$, are the solutions to the system

$$\begin{aligned} y + 2\lambda x &= 0 \\ x + 2\lambda y &= 0 \\ \text{and } x^2 + y^2 &= 4 \end{aligned}$$

Notice that if $x = 0$ then for the first equation we would need $y = 0$ as well, but $(0,0)$ does not satisfy the constraint, so x cannot be 0. Likewise, for $y = 0$ the second equation gives $x = 0$ as well, so y also cannot be 0. Therefore we can rearrange the first two equations to

$$\lambda = \frac{-y}{2x} \quad \text{and} \quad \lambda = \frac{-x}{2y}$$

Therefore we need

$$-\frac{y}{2x} = -\frac{x}{2y} \quad \Rightarrow \quad 2y^2 = 2x^2 \quad \Rightarrow \quad y^2 = x^2$$

Using this in the third equation in the system we need $x^2 + x^2 = 4$, i.e. $2x^2 = 4$ so that $x^2 = 2$. Thus we need $x = \pm\sqrt{2}$, and we also need $y^2 = x^2$, so that y may be either x or $-x$. This gives 4 candidate points: $(-\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2}), (\sqrt{2}, -\sqrt{2})$ and $(\sqrt{2}, \sqrt{2})$. When x and y have different signs, $f(x,y) = xy$ is negative, whereas when x and y have the same sign, $f(x,y)$ is positive, so this positive value will be the constrained maximum of f and the negative value will be the constrained minimum. That is, we see that $f(-\sqrt{2}, -\sqrt{2}) = f(\sqrt{2}, \sqrt{2}) = 2$ while $f(-\sqrt{2}, \sqrt{2}) = f(-\sqrt{2}, \sqrt{2}) = -2$, so the maximum value of f subject to the given constraint is 2 (and the minimum is -2).

A23. If $f(x) = \tan x$, what is $f'(x)$?

A: $\dfrac{\sin x}{\cos x}$	B: $\dfrac{\cos x}{\sin x}$	C: $-\dfrac{\cos x}{\sin x}$	D: $\sec^2 x$	E: $\sec x \tan x$

Solution:

$$f'(x) = \frac{d}{dx}(\tan x) = \sec^2 x$$

Or if you don't remember the derivative of $\tan x$, then expressing $f(x)$ as $\dfrac{\sin x}{\cos x}$ gives

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\left[\frac{d}{dx}(\sin x)\right](\cos x) - (\sin x)\left[\frac{d}{dx}(\cos x)\right]}{(\cos x)^2} \\ &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} \end{aligned}$$

And since we know that for any angle x, $\sin^2 x + \cos^2 x = 1$ we have

$$f'(x) = \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2 = (\sec x)^2 = \sec^2 x$$

A24. Find y' if $y = \sin^2 x + \cos x$.

A: $2\sin x\cos x - \sin x$	B: $2\sin x\cos x + \sin x$	C: $2\sin x + \cos(1)$	D: $\sin x$	E: $3\sin x$

Solution:

$$y' = \frac{d}{dx}(\sin x)^2 + \frac{d}{dx}(\cos x) = 2(\sin x)^1\left[\frac{d}{dx}(\sin x)\right] + (-\sin x) = 2\sin x\cos x - \sin x$$

A25. Find $f'(x)$ for the function $f(x) = \sin(x^2e^x)$.

A: $\sin(2xe^x)$	B: $e^x(2x + x^2)\cos(x^2e^x)$	C: $2xe^x\cos(x^2e^x)$
D: $\cos[e^x(2x + x^2)]$	E: $e^x(2x + x^2)\sin(x^2e^x)$	

Solution: We need the chain rule:

$$\begin{aligned} f'(x) &= \frac{d}{dx}[\sin(x^2e^x)] = [\cos(x^2e^x)]\left[\frac{d}{dx}(x^2e^x)\right] = [\cos(x^2e^x)][2x(e^x) + x^2(e^x)] \\ &= (2xe^x + x^2e^x)\cos(x^2e^x) = e^x(2x + x^2)\cos(x^2e^x) \end{aligned}$$

A26. Find $\displaystyle\int e^t \sin t\, dt$.

A: $\dfrac{e^t(\sin t - \cos t)}{2} + C$	B: $\dfrac{e^t(\cos t - \sin t)}{2} + C$	C: $-e^t\cos t + C$
D: 0	E: Cannot be determined.	

Solution: We need integration by parts. In this case, there is no good way for deciding which term in the product $e^t \sin t$ should be u and which should be v', so we can use either term as u. Since the derivative of $\sin x$ does not involve a negative, but the antiderivative does, perhaps you would prefer using $u = \sin t$ so that $dv = e^t\, dt$. This gives $du = \cos t\, dt$ and $v = e^t$, so we get

$$\int e^t \sin t\, dt = \int u\, dv = uv - \int v\, du = (\sin t)(e^t) - \int (e^t)(\cos t)\, dt = e^t \sin t - \int e^t \cos t\, dt$$

For $\displaystyle\int e^t \cos t\, dt$ we need integration by parts again. If we choose $dv = \cos t\, dt$, this will just undo what we have already done, so we need $u = \cos t$ and $dv = e^t\, dt$. (That is, we need to make choices similar to those we made the first time.) This gives $du = -\sin t\, dt$ and $v = e^t$ so we get

$$\int e^t \cos t\, dt = e^t \cos t - \int e^t(-\sin t)\, dt = e^t \cos t + \int e^t \sin t\, dt$$

Putting this into what we had after the first integration by parts, we have

$$\int e^t \sin t\, dt = e^t \sin t - \int e^t \cos t = e^t \sin t - \left(e^t \cos t + \int e^t \sin t\, dt\right) = e^t \sin t - e^t \cos t - \int e^t \sin t\, dt$$

Rearranging this by adding $\displaystyle\int e^t \sin t\, dt$ to both sides, and adding the integration constant, because we won't be integrating anymore, we get

$$2\int e^t \sin t\, dt = e^t \sin t - e^t \cos t + C = e^t(\sin t - \cos t) + C$$

and so we see that

$$\int e^t \sin t\, dt = \frac{e^t(\sin t - \cos t)}{2} + C$$

A27. Find the general solution to the differential equation $\dfrac{dy}{dx} = \dfrac{e^x}{y^2}$.

A: $y(x) = 3e^x + C$	B: $y(x) = \sqrt{e^x + C}$	C: $y(x) = \sqrt[3]{3e^x} + C$
D: $y(x) = \sqrt[3]{e^x + C}$	E: $y(x) = \sqrt[3]{3e^x + C}$	

Solution: We have a separable first order differential equation, so we separate and then integrate:

$$\frac{dy}{dx} = \frac{e^x}{y^2} \quad \Rightarrow \quad y^2\,dy = e^x\,dx$$
$$\Rightarrow \quad \int y^2\,dy = \int e^x\,dx$$
$$\Rightarrow \quad \frac{y^3}{3} = e^x + C$$
$$\Rightarrow \quad y^3 = 3e^x + C$$
$$\Rightarrow \quad y = \sqrt[3]{3e^x + C}$$

A28. If $\dfrac{dy}{dx} = e^{2x-y}$ and $y(\ln 2) = \ln 4$, what is $y(\ln 4)$?

A: $\ln 2$	B: $\ln 4$	C: $\ln 6$	D: $\ln 10$	E: $\ln 16$

Solution: We have $\dfrac{dy}{dx} = e^{2x-y} = \dfrac{e^{2x}}{e^y}$ so we have a separable first order differential equation. Separating and integrating we have

$$\frac{dy}{dx} = \frac{e^{2x}}{e^y} \quad \Rightarrow \quad \int e^y\,dy = \int e^{2x}dx \quad \Rightarrow \quad e^y = \frac{e^{2x}}{2} + C$$

The fact that $y(\ln 2) = \ln 4$ tells us that when $x = \ln 2$, $y = \ln 4$. Using this in $e^y = \dfrac{e^{2x}}{2} + C$ we get

$$e^{\ln 4} = \frac{e^{2\ln 2}}{2} + C \quad \Rightarrow \quad 4 = \frac{e^{\ln 2^2}}{2} + C \quad \Rightarrow 4 = \frac{4}{2} + C \quad \Rightarrow 4 = 2 + C$$

so we see that $C = 2$. Therefore $e^y = \dfrac{e^{2x}}{2} + 2$ which gives

$$\ln e^y = \ln\left(\frac{e^{2x}}{2} + 2\right) \quad \Rightarrow \quad y = \ln\left(\frac{e^{2x}}{2} + 2\right)$$

Therefore $y(x) = \ln\left(\dfrac{e^{2x}}{2} + 2\right)$ and so

$$y(\ln 4) = \ln\left(\frac{e^{2\ln 4}}{2} + 2\right) = \ln\left(\frac{e^{\ln 4^2}}{2} + 2\right) = \ln\left(\frac{4^2}{2} + 2\right) = \ln(8 + 2) = \ln 10$$

A29. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{\sin x}{\cos y}$.

A: $y = C + \ln\left\lvert\sec\left(\dfrac{x}{y}\right)\right\rvert$	B: $\sin y = C + \cos x$	C: $\sin y = C - \cos x$
D: $\cos y = C + \sin x$	E: $\cos y = C - \sin x$	

Solution: Again, this is a separable first order DE. We get

$$\frac{dy}{dx} = \frac{\sin x}{\cos y} \quad \Rightarrow \quad \int \cos y\,dy = \int \sin x\,dx \quad \Rightarrow \sin y = -\cos x + C = C - \cos x$$

A30. Which of the following is an integrating factor that can be used in solving the linear differential equation $\dfrac{dy}{dx} + \dfrac{3y}{x} = 2\sec^2 x$?

A: x^3	B: $e^{(x^3)}$	C: $e^{3/x}$	D: $3\ln x$	E: $e^{2\tan x}$

Solution: We recognize that we have the form $\dfrac{dy}{dx} + yP(x) = Q(x)$ with $P(x) = \dfrac{3}{x}$ and $Q(x) = 2\sec^2 x$. An integrating factor which can be used to solve this linear first order DE is $u(x) = e^{\int P(x)\,dx}$, using any antiderivative of $P(x)$. For $P(x) = \frac{3}{x} = 3\left(\frac{1}{x}\right)$, we know that $\ln x$ is *an* antiderivative of $\frac{1}{x}$ (although not the most general one), so we can use $3\ln x$ as an antiderivative of $\frac{3}{x}$. This gives the integrating factor as

$$u(x) = e^{\int P(x)\,dx} = e^{\int \frac{3}{x}\,dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

A31. Solve the differential equation $\dfrac{dy}{dx} + 2y = xe^{-x}$ subject to the condition that $y(0) = 1$, using the integrating factor e^{2x}.

A: $y(x) = (x-1)e^x + 2$	B: $y(x) = \dfrac{x-1}{e^x} + 2$	C: $y(x) = \dfrac{x-1}{e^x} + \dfrac{2}{e^{2x}}$
D: $y(x) = \dfrac{1}{3e^x}\left(\dfrac{1}{3} - x\right) + \dfrac{10}{9}$	E: $y(x) = \dfrac{1}{3e^x}\left(\dfrac{1}{3} - x\right) + \dfrac{10}{9e^{2x}}$	

Solution: We know that for the linear first order DE $\dfrac{dy}{dx} + yP(x) = Q(x)$, using the integrating factor $u(x)$ the general solution is given by

$$y\,u(x) = \int Q(x)\,u(x)\,dx \qquad \Rightarrow \qquad y = \frac{1}{u(x)}\int Q(x)\,u(x)\,dx$$

In this case we have $P(x) = 2$ and $Q(x) = xe^{-x}$, and we are to use the integrating factor $u(x) = e^{2x}$ (which of course is $e^{\int P(x)\,dx}$). Therefore we get

$$ye^{2x} = \int (xe^{-x})(e^{2x})\,dx \qquad \Rightarrow \qquad y = e^{-2x}\int xe^{-x+2x}dx = e^{-2x}\int xe^x\,dx$$

For this integral we need integration by parts. Letting $u = x$ and $dv = e^x dx$ we have $du = dx$ and $v = e^x$ so we get

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C$$

Therefore the general solution to the DE is

$$y = e^{-2x}\int xe^x dx = e^{-2x}[e^x(x-1) + C] = \frac{e^x(x-1) + C}{e^{2x}} = \frac{x-1}{e^x} + \frac{C}{e^{2x}}$$

And now using the fact that $y(0) = 1$, i.e. when $x = 0$, $y = 1$, we have

$$y = \frac{x-1}{e^x} + \frac{C}{e^{2x}} \qquad \Rightarrow 1 = \frac{0-1}{e^0} + \frac{C}{e^{2(0)}} = \frac{-1}{1} + \frac{C}{1} = -1 + C$$

For $1 = -1 + C$ we need $C = 2$, so the particular solution to the given initial value problem is

$$y(x) = \frac{x-1}{e^x} + \frac{2}{e^{2x}}$$

A32. The differential equation expressed in Newton's Law of Cooling, $\frac{dy}{dt} = k(y - M)$, can be considered as either a separable differential equation or a linear differential equation. Approached as a linear differential equation, what integrating factor should be used?

A: e^{kt}	B: e^{-kt}	C: e^{ky}	D: e^{-ky}	E: e^{-kM}

Solution:

$$\frac{dy}{dt} = k(y - M) = ky - kM \qquad \Rightarrow \qquad \frac{dy}{dt} - ky = -kM$$

We see that in the linear first order DE we have $P(t) = -k$ and $Q(t) = -kM$. Therefore the integrating factor is

$$u(x) = e^{\int P(t)\,dt} = e^{\int (-k)\,dt} = e^{-kt}$$

A33. If $\frac{dy}{dt} = ky$, $y(0) = 5$ and $y(1) = 10$, find $y(2)$.

A: 5	B: 10	C: 15	D: 20	E: $\ln 2$

Solution: We know that the solution to $\frac{dy}{dt} = ky$ is $y(t) = y_0 e^{kt}$, where $y_0 = y(0)$. In this case we are told that $y(0) = 5$, so we have $y(t) = 5e^{kt}$. We use the fact that $y(1) = 10$ to find the value of k:

$$y(1) = 5e^k = 10 \qquad \Rightarrow \qquad e^k = \frac{10}{5} = 2 \qquad \Rightarrow \qquad k = \ln 2$$

Therefore the solution to $\frac{dy}{dt} = ky$ with $y(0) = 5$ and $y(1) = 10$ is $y(t) = 5e^{t \ln 2}$ and so we get

$$y(2) = 5e^{2 \ln 2} = 5e^{\ln 2^2} = 5(2^2) = 20$$

A34. If a certain radioactive element has a half-life of 100 years and there are 15 grams of this element now, how long from now will there be only 5 grams remaining?

A: $\frac{100 \ln 2}{\ln 3}$ years	B: $\frac{\ln 2}{100 \ln 3}$ years	C: $\frac{100 \ln 3}{\ln 2}$ years	D: $\frac{\ln 3}{100 \ln 2}$ years	E: $\frac{4}{3}$ centuries

Solution: We know that radioactive elements exhibit exponential decay, i.e. decay according to the DE $\frac{dy}{dt} = ky$ for some negative constant k, where $y(t)$ is the quantity of the element present at time t. And we know that the solution to this DE is $y(t) = y_0 e^{kt}$, where $y_0 = y(0)$. Since there are 15 grams of the element now, i.e. at $t = 0$, we have $y(t) = 15e^{kt}$. The fact that the element has a half-life of 100 years tells us that at time $t = 100$ there will only be half as much of the element as there is now, so $y(100) = 7.5$. We use this to find the value of k:

$$y(100) = 15e^{100k} = 7.5 \qquad \Rightarrow \qquad e^{100k} = \frac{7.5}{15} = \frac{1}{2} \qquad \Rightarrow \qquad 100k = \ln \frac{1}{2} = \ln 1 - \ln 2 = -\ln 2 \qquad \Rightarrow k = -\frac{\ln 2}{100}$$

(Or maybe you just remembered that in radioactive decay, $k = -\frac{\ln 2}{\text{half-life}}$.)

Therefore the quantity of the element at time t is given by $y(t) = 15e^{-t(\ln 2)/100}$. We need to find the value of t at which $y(t) = 5$. We get

$$15e^{-t(\ln 2)/100} = 5 \;\Rightarrow\; e^{-t(\ln 2)/100} = \frac{5}{15} = \frac{1}{3} \;\Rightarrow\; -\frac{t \ln 2}{100} = \ln \frac{1}{3} = -\ln 3 \;\Rightarrow\; t = (\ln 3)\left(\frac{100}{\ln 2}\right) = \frac{100 \ln 3}{\ln 2}$$

That is, there will be only 5 grams of the element remaining after $\frac{100 \ln 3}{\ln 2}$ years.

A35. A tank contains 100 litres of pure water. At time $t = 0$, two valves are opened. One valve allows a brine solution containing .2 kg of salt per litre to enter the tank at a rate of 3 litres per minute. The other valve allows the well-mixed solution to exit the tank, also at a rate of 3 litres per minute. Which one of the following is the mathematical model that would be solved to find a formula for the amount of salt in the tank, measured in kilograms, t minutes after the valves are opened?

A: $\frac{dy}{dt} = .6$, $\frac{dy}{dt} = -.03y$, $y(0) = 0$	B: $\frac{dy}{dt} = .6 - .3y$, $y(0) = 0$	C: $\frac{dy}{dt} = .3 - .06y$, $y(0) = 100$
D: $\frac{dy}{dt} = .6 - .03y$, $y(0) = 100$	E: $\frac{dy}{dt} = .6 - .03y$, $y(0) = 0$	

Solution: Let $y(t)$ be the number of kilograms of salt in the tank t minutes after the valves are opened. Since there is no salt in the tank initially, i.e. at time $t = 0$, we know that $y(0) = 0$. And the rate of change in y, with respect to t is given by

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}$$

where rate in and rate out refer to the rate at which salt is entering and leaving the tank.

We have brine entering the tank at rate 3 litres per minute, and this brine contains .2 kg per litre, so each minute, $3(.2) = .6$ kg of salt enter the tank. That is, we have

$$\text{rate in } = .6 \text{ kg / minute}$$

And when there are y kg of salt in the tank, since the solution in the tank is well-mixed and there are always 100 litres in the tank, this salt is evenly spread throughout the liquid in the tank. Therefore each litre of liquid (i.e. water) in the tank contains $\frac{y}{100}$ kg of salt. This liquid is leaving the tank at a rate of 3 litres per minute, so the rate at which salt is leaving the tank is

$$\text{rate out} = \frac{y}{100} \text{ kg / litre} \times 3 \text{ litres / minute } = \frac{3y}{100} = .03y \text{ kg / minute}$$

Therefore the differential equation giving the rate of change in the amount of salt in the tank is

$$\frac{dy}{dt} = \text{rate in} - \text{ rate out} = .6 - .03y$$

This DE, together with the initial condition that $y(0) = 0$, gives the mathematical model describing the situation, i.e. the model which would need to be solved to find $y(t)$.

PART B (15 marks)

3 marks B1. Find $\int_1^e x^{-2} \ln x \, dx$. Express your answer as simply as possible.

Solution: We need integration by parts. Let $u = \ln x$ and $dv = x^{-2}dx$. Then $du = \frac{1}{x}$ and $v = \frac{x^{-1}}{-1} = -\frac{1}{x}$, so we have

$$\int x^{-2} \ln x \, dx = \int u \, dv = uv - \int v \, du = (\ln x)\left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right) dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx$$

$$= -\frac{\ln x}{x} + \int x^{-2} dx = -\frac{\ln x}{x} + \frac{x^{-1}}{-1} + C = -\frac{\ln x}{x} - \frac{1}{x} + C$$

That is, we see that $-\frac{\ln x}{x} - \frac{1}{x}$ is an antiderivative of $x^{-2} \ln x$ so we get

$$\int_1^e x^{-2} \ln x \, dx = \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^e = \left(-\frac{\ln e}{e} - \frac{1}{e}\right) - \left(\frac{-\ln 1}{1} - \frac{1}{1}\right) = -\frac{1}{e} - \frac{1}{e} + \frac{0}{1} + 1 = 1 - \frac{2}{e}$$

3 marks B2. Find $\displaystyle\int_0^{\pi/3} \frac{\sin x}{\cos x}\,dx$.

Solution: You may remember that $\ln|\sec x|$ is an antiderivative of $\tan x = \dfrac{\sin x}{\cos x}$. If so, you can find the value of the definite integral directly:

$$\begin{aligned}\int_0^{\pi/3} \frac{\sin x}{\cos x}\,dx &= \int_0^{\pi/3} \tan x\,dx = [\ln|\sec x|]_0^{\pi/3} = \left[\ln\left|\frac{1}{\cos x}\right|\right]_0^{\pi/3} \\ &= \ln\left|\frac{1}{\cos\frac{\pi}{3}}\right| - \ln\left|\frac{1}{\cos 0}\right| = \ln\left(\frac{1}{1/2}\right) - \ln\left(\frac{1}{1}\right) = \ln 2 - \ln 1 = \ln 2\end{aligned}$$

On the other hand, if you don't remember an antiderivative of $\tan x$, you can use the substitution rule. Let $u = \cos x$ so that $du = -\sin x\,dx$ and therefore $\sin x\,dx = -du$. When $x = 0$ we have $u = \cos 0 = 1$ and when $x = \frac{\pi}{3}$ we have $u = \cos\frac{\pi}{3} = \frac{1}{2}$ so we get

$$\int_0^{\pi/3} \frac{\sin x}{\cos x}\,dx = -\int_1^{1/2} \left(\frac{1}{u}\right)\,du = \int_{1/2}^1 \frac{1}{u}\,du = [\ln u]_{1/2}^1 = \ln 1 - \ln\frac{1}{2} = 0 - (\ln 1 - \ln 2) = \ln 2$$

4 marks B3. Consider the problem of using the method of Lagrange multipliers to find the maximum or minimum of $f(x,y) = 2x + 4y$ subject to the constraint $x^2 + y^2 = 5$.

(a) What system of equations has to be solved?

Solution: The Lagrange function is $F(x,y,\lambda) = 2x + 4y + \lambda(x^2 + y^2 - 5)$. The system of equations which must be solved is the system in which each of the first partials of F is set equal to 0. We get

$$\begin{aligned}F_x(x,y,\lambda) &= \frac{\partial}{\partial x}[2x + 4y + \lambda(x^2 + y^2 - 5)] = 2 + 0 + \lambda(2x + 0 - 0) = 2 + 2x\lambda \\ F_y(x,y,\lambda) &= \frac{\partial}{\partial y}[2x + 4y + \lambda(x^2 + y^2 - 5)] = 0 + 4 + \lambda(0 + 2y - 0) = 4 + 2y\lambda \\ F_\lambda(x,y,\lambda) &= \frac{\partial}{\partial \lambda}[2x + 4y + \lambda(x^2 + y^2 - 5)] = 0 + 0 + 1(x^2 + y^2 - 5) = x^2 + y^2 - 5\end{aligned}$$

Therefore the system of equations that needs to be solved is

$$\begin{aligned}2 + 2x\lambda &= 0 \\ 4 + 2y\lambda &= 0 \\ x^2 + y^2 - 5 &= 0\end{aligned}$$

(Of course, the last equation in the system could be expressed as $x^2 + y^2 = 5$.)

(b) Find all points at which the maximum or minimum of this constrained optimization problem could occur.

Solution: The points at which the maximum or minimum of the constrained optimization problem could occur are the points which satisfy the system of equations found above. That is, here we are asked to find all solutions to the system found in part (a). Notice that from the first 2 equations, neither x nor y can be 0. Rearranging the first equation in the system we get

$$2 + 2x\lambda = 0 \quad\Rightarrow\quad 2x\lambda = -2 \quad\Rightarrow\quad \lambda = \frac{-2}{2x} = -\frac{1}{x}$$

Similarly, rearranging the second equation we have

$$4 + 2y\lambda = 0 \quad \Rightarrow \quad 2y\lambda = -4 \quad \Rightarrow \quad \lambda = \frac{-4}{2y} = -\frac{2}{y}$$

Equating these two expressions for λ we get

$$-\frac{1}{x} = -\frac{2}{y} \quad \Rightarrow \quad y = 2x$$

And now substituting this into the third equation in the system we have

$$x^2 + y^2 - 5 = 0 \quad \Rightarrow \quad x^2 + (2x)^2 = 5 \quad \Rightarrow \quad x^2 + 4x^2 = 5 \quad \Rightarrow \quad 5x^2 = 5 \quad \Rightarrow \quad x^2 = 1$$

Therefore we need $x = \pm 1$, with $y = 2x$. This gives 2 candidate points: $(x, y) = (-1, -2)$ and $(x, y) = (1, 2)$.

(c) What is the maximum of f, subject to the given constraint?

Solution: We test each of the candidate points in the function $f(x, y) = 2x + 4y$ to see which gives the larger value:

$$f(-1, -2) = 2(-1) + 4(-2) = -2 - 8 = -10 \quad \text{and} \quad f(1, 2) = 2(1) + 4(2) = 2 + 8 = 10$$

We see that the maximum of f, subject to the given constraint, is 10 (which is obtained at $(x, y) = (1, 2)$).

2 marks B4. The slope of the tangent line to a certain curve at any point (x, y) on the curve is given by $2[\cos(2\sin x)]\cos x$. If the curve passes through the point $(x, y) = (0, -1)$, find an equation of the curve.

Solution: Since the slope of the tangent line to a curve in xy-space is given by $\frac{dy}{dx}$, we have the first order DE

$$\frac{dy}{dx} = 2[\cos(2\sin x)]\cos x$$

We can treat this as a separable DE:

$$dy = 2[\cos(2\sin x)]\cos x\,dx \quad \Rightarrow \quad \int dy = \int 2[\cos(2\sin x)]\cos x\,dx \quad \Rightarrow \quad y = \int 2[\cos(2\sin x)]\cos x\,dx$$

For this integral we need the substitution $u = 2\sin x$ (i.e. let u equal the angle in the innermost trig function), which gives $du = 2\cos x\,dx$. We get

$$y = \int 2[\cos(2\sin x)]\cos x\,dx = \int [\cos(2\sin x)](2\cos x)\,dx = \int \cos u\,du = \sin u + C = \sin(2\sin x) + C$$

Now, we use the fact that the curve passes through the point $(x, y) = (0, -1)$ to solve for C. That is, we know that $x = 0$ and $y = -1$ must satisfy the equation. This gives

$$y = \sin(2\sin x) + C \quad \Rightarrow \quad -1 = \sin(2\sin 0) + C = \sin 2(0) + C = \sin 0 + C = 0 + C = C$$

Therefore we need $C = -1$ so an equation of the curve is

$$y = \sin(2\sin x) - 1$$

3 marks B5. At any time t, the growth of a certain population is proportional to the size of the population. When first observed, there were 1,000 members in the population. At a time 10 time units later, the population had grown to 1200 members.
A mathematical model for this situation is:

$$\begin{aligned} \frac{dP}{dt} &= kP \\ P(0) &= 1000 \\ P(10) &= 1200 \end{aligned}$$

(a) Find a formula for $P(t)$, the size of the population t time units after the initial observation.

Solution: This is an instance of exponential growth. We know that the solution to $\frac{dP}{dt} = kP$ is $P(t) = P_0 e^{kt}$, where $P_0 = P(0)$. Since we are told that $P(0) = 1000$, we have $P(t) = 1000e^{kt}$. We need to find the value of k, using the other observation of the size of the population. Since $P(10) = 1000e^{10k}$ we need

$$1000e^{10k} = 1200 \quad \Rightarrow \quad e^{10k} = \frac{1200}{1000} = \frac{12}{10} \quad \Rightarrow \quad 10k = \ln\frac{12}{10} \quad \Rightarrow \quad k = \frac{\ln\frac{12}{10}}{10}$$

This gives the formula for $P(t)$ as

$$P(t) = 1000e^{t[\ln(12/10)]/10]}$$

(b) Find the size of the population 30 time units after the initial observation. Express your answer as simply as possible.

Solution: Using the formula found above we get

$$P(30) = 1000e^{30[\ln(12/10)]/10} = 1000e^{3\ln(12/10)} = 1000e^{\ln(12/10)^3} = 1000\left(\frac{12}{10}\right)^3 = 12^3 = 1728$$

(*Note:* The final answer did not have to be expressed as 1728. $(12)^3$ and $1000(1.2)^3$ and similar representations also received the mark in part (b). Any correct form of the answer that didn't still involve e and ln was accepted.)