Week 3 Practice Problems MATH 203

 # 1. The function f is given by the formula

$$f(x) = \begin{cases} \frac{6x^3+23x^2+6x+40}{x+4} & \text{if } x < -4; \\ \\ 4x^2 + 3x + a & x \geq -4 \end{cases}$$

Find the value of a so that $\lim_{x\to -4} f(x)$ exists? (i.e continuous at -4)

 # 2. Let

$$f(x) = \frac{2x^2 + 3x - 65}{x - 5}$$

Show that $f(x)$ has a removable discontinuity at $x = 5$ and determine what value for $f(5)$ would make $f(x)$ continuous at $x = 5$.

 # 3.

(a) If $f(x) = x^3 - 5x + 1$, find $f'(1)$

(b) Find the equation of the tangent line to the curve $f(x)$ at the point $(1, -3)$

 # 4. Let

$$f(x) = -3x^3 + 4x + 1$$

Use the definition of a derivative to calculate $f'(x)$ and $f''(x)$

Final Exam April 2015 # 3.

Find all horizontal and vertical asymptotes of the function

$$f(x) = \frac{|x|\sqrt{4x^2+1} - 2x^2}{x^2 - 3}$$

Solutions:.

 # 1.

For $\lim_{x\to -4} f(x)$ to exist, then

$$\lim_{x\to -4} f(x) = \lim_{x\to -4^-} f(x) = \lim_{x\to -4^+} f(x)$$

$$\lim_{x\to -4^-} \frac{6x^3+23x^2+6x+40}{x+4} = \lim_{x\to -4^+} 4x^2+3x+a$$
$$\lim_{x\to -4} \frac{6x^3+23x^2+6x+40}{x+4} = \lim_{x\to -4} 4x^2+3x+a$$

Compute

$$\lim_{x\to -4^-} \frac{6x^3+23x^2+6x+40}{x+4}$$

In class, we used long division to simplify the rational function. We may use synthetic division as well,

$$\begin{array}{r|rrrr} & 6 & 23 & 6 & 40 \\ -4 & & -24 & 4 & -40 \\ \hline & 6 & -1 & 10 & 0 \end{array}$$

$\{6, -1, 10, 0\}$ are the coefficients of x^2, x, 1, $1/x$, respectively. Thus,

$$\lim_{x\to -4^-} \frac{6x^3-23x^2+6x-40}{x+4} = \lim_{x\to -4^-} \frac{(6x^2-x+10)(x+4)}{x+4} = \lim_{x\to -4^-} 6x^2-x+10$$

$$= 6(-4)^2-(-4)+10 = 110$$

If $(-4, f(-4))$ is a point where $f(x)$ is continuous, then we get

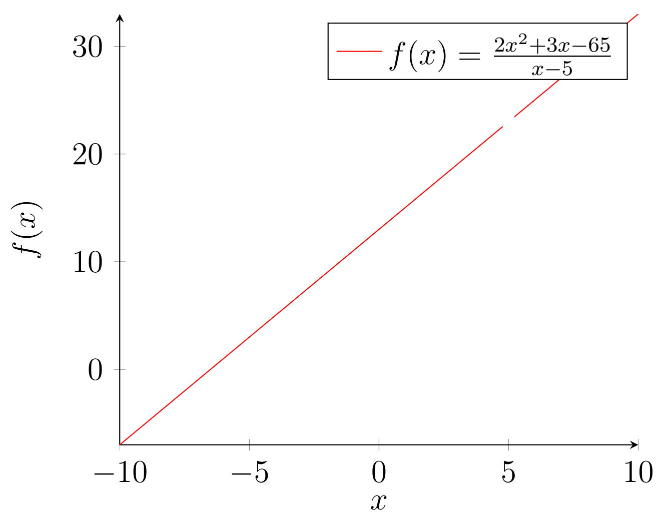
$$\begin{aligned}\lim_{x\to -4^+} f(x) &= \lim_{x\to -4^-} f(x)\\ \lim_{x\to -4^+} 4x^2 + 3x + a &= 110\\ 4(-4)^2 + 3(-4) + a &= 110\\ a &= 58\end{aligned}$$

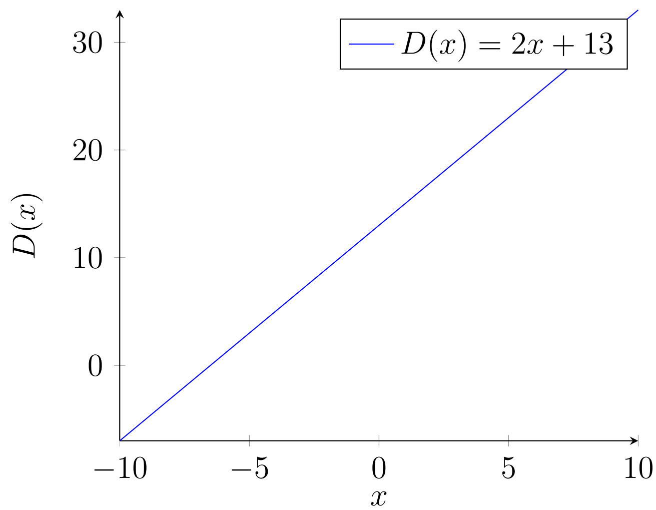
2.

Remark: Suppose $f(x) = \frac{P(x)}{Q(x)}$ is rational function. If $P(x) \div Q(x) = D(x)$ without remainder, then the graph of $f(x)$ is $D(x)$ where $x = a$ is not defined in $f(x)$ whenever $Q(a) = 0$.

$$\begin{aligned}\lim_{x\to 5} f(x) = \lim_{x\to 5} \frac{2x^2 + 3x - 65}{x - 5} &= \lim_{x\to 5} \frac{(2x + 13)(x - 5)}{x - 5}\\ = \lim_{x\to 5} \underbrace{2x + 13}_{=D(x)} &= 2(5) + 13 = 23\end{aligned}$$

Consider $D(x) = 2x + 13$ and observe the difference between the graph of $f(x)$ and $D(x)$.





The two graphs are exactly alike except when $x = 5$, since $D(5) = 23$ but $f(5)$ is undefined. (i.e, there is a hole on the line $f(x)$ at the point $(5, 23)$). Hence, $f(x)$ has a removable discontinuity at $x = 5$ and $f(5)$ has to be equal to 23 to make $f(x)$ continuous.

 # 3(a). There are two ways to compute the derivative $f'(x)$,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Let's use the right hand side of the equality to solve $f'(1)$. We are given $f(x) = x^3 - 5x + 1$, then $f(1) = -3$ and

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{(x^3 - 5x + 1) - (-3)}{x - 1} = \lim_{x \to 1} \frac{x^3 - 5x + 4}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x^2 + x - 4)(x - 1)}{x - 1} = \lim_{x \to 1} x^2 + x - 4 = (1)^2 + 1 - 4 = -2$$

 # 3(b). The slope of the tangent line $y = mx + b$ is the derivative of $f(x)$ at point $x = 1$. (i.e, $m = f'(1) = -2$). The equation of the tangent line is

$$y = -2x + b$$

and passes through the point $(1, f(1)) = (1, -3)$. Hence, for $(x, y) = (1, -3)$

$$y = -2x + b \Rightarrow -3 = -2(1) + b \;\Rightarrow\; b = -1$$

$$y = -2x - 1$$

 # 4. For this question, we will use the other definition of derivative,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

to compute $f'(x)$ for $f(x) = -3x^3 + 4x + 1$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(-3(x+h)^3 + 4(x+h) + 1) - (-3x^3 + 4x + 1)}{h}$$

$$= \lim_{h\to 0} \frac{-3(x^3 + 3x^2h + 3xh^2 + h^3) + 4x + 4h + 1 + 3x^3 - 4x - 1}{h}$$

$$= \lim_{h\to 0} \frac{-3x^3 - 9x^2h - 9xh^2 - 3h^3 + 4x + 4h + 1 + 3x^3 - 4x - 1}{h}$$

$$= \lim_{h\to 0} \frac{-9x^2h - 9xh^2 - 3h^3 + 4h}{h} = \lim_{h\to 0} -9x^2 - 9xh - 3h^2 + 4$$

Clearly, $-9xh$ and $-3h^2$ goes to zero as h approaches 0. Therefore,

$$f'(x) = \lim_{h\to 0} -9x^2 - 9xh - 3h^2 + 4 = -9x^2 + 4$$

For the second derivative,

$$f"(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{-9(x+h)^2 + 4 - (-9x^2 + 4)}{h}$$

$$= \lim_{h\to 0} \frac{-9(x^2 + 2xh + h^2) + 4 + 9x^2 - 4)}{h} = \lim_{h\to 0} \frac{-9x^2 - 18xh - 9h^2 + 4 + 9x^2 - 4}{h}$$

$$= \lim_{h\to 0} \frac{-18xh - 9h^2}{h} = \lim_{h\to 0} -18x - 9h = -18x$$