COMPLEX NUMBERS

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What is a Number? Numbers are "symbols" used for measuring and counting purposes. Examples of numbers are:

- Natural Numbers: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$;
- Integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$;
- Rationals: $\mathbb{Q} = \{\frac{a}{b} \mid a \text{ and } b \text{ are integers}\}$.

All these numbers are part of a larger collection of such symbols, the **Real Numbers**, denoted by $\mathbb{R}$.

Are There More Numbers? From Calculus we know that there is another collection of numbers, the irrationals. We can say, in short, that the irrationals are the real numbers that are not rationals. For a more precise definition we need to consider limits, which we will avoid since it is not part of this course. Examples of irrational numbers are: $\sqrt{2}$, π, e, etc.

In general, numbers have been "obtained" by considering the solution set of a given type of equations. For example, the integers are obtained from equations of the form

$$x + a = 0, \qquad a \in \mathbb{N},$$

or the rationals, are obtained from equations of the form

$$ax = b, \qquad a, b \in \mathbb{Z}.$$

In this course we will consider two extensions of the real numbers: the complex numbers and matrices.

1. Complex Numbers

The complex numbers **are** the solutions of equations of the form

$$x^2 = a, \qquad a \in \mathbb{R}.$$

Fortunately, we only need to be concerned with the solution of $x^2 = -1$.

Definition. The symbol

$$\imath = \sqrt{-1}$$

will be called the **imaginary unit**. Notice that $\imath^2 = -1$.

Definition. The **complex numbers**, denoted by $\mathbb{C}$, are defined as the collection of ALL the expressions of the form

$$a + b\,\imath, \qquad a, b \in \mathbb{R}.$$

Remark. Any real number a can regarded as the complex number $a+0\,\imath$.

Usually we will use the letters z and w to denote complex numbers.

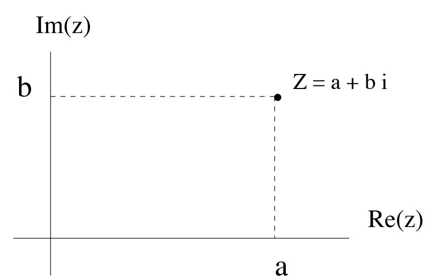
Definition. Given the complex number $z = a + b\,\imath$, the **real part** of z is given by $Re(z) = a$. Similarly, the **imaginary part** of z is given by $Im(z) = b$.

Remark. Given a complex number z, if $Im(z) = 0$ then z is in fact a real number. If $Re(z) = 0$ then we say that z is **pure imaginary**.

In order to manipulate the complex numbers we need to establish what does it mean for two complex numbers to be "equal".

Definition. Two complex numbers z and w are **equal** if and only if $Re(z) = Re(w)$ AND $Im(z) = Im(w)$.

With this definition we have that for every complex number z there is a UNIQUE ordered pair $(Re(z), Im(z))$. Thus, we can represent the complex numbers by means of a coordinate system:



1.1. **Basic Operations.** We can regard the complex numbers as "vectors", but we can also consider them as "polynomials evaluated at $\imath$", once we accept that $\imath$ is a symbol.

All of this allows us to define the basic operation on the complex numbers:

Definition. Let $z = a + b\,\imath$ and $w = c + d\,\imath$ be two complex numbers.

- Addition: $z + w = (a + c) \; + \; (b + d)\imath$.
- Subtraction: $z - w = (a - c) \; + \; (b - d)\imath$.
- Product: $z \cdot w = (ac - bd) \; + \; (ad + cb)\imath$.

Example. Consider the complex numbers $z = 1 + 3\imath$ and $w = 2 - \imath$. Then

$$z + w = (1 + 3\imath) + (2 - \imath) = 3 + 2\imath.$$

$$z - w = (1 + 3\imath) - (2 - \imath) = -1 + 4\imath.$$

$$z \cdot w = (1 + 3\imath)(2 - \imath) = 2 - \imath + 6\imath - 3\imath^2 = 5 + 5\imath.$$

In order to define $\frac{z}{w}$ we need to introduce some additional concepts.

Definition. The **conjugate** of a complex number $z = a + b\,\imath$ is the complex number

$$\bar{z} = a - b\,\imath.$$

Notice that $\bar{z}$ is the reflexion of z along the Re-axis.

Examples.

$$\overline{z} = \overline{1 + 3\imath} = 1 - 3\imath.$$

$$\overline{w} = \overline{2 - \imath} = 2 + \imath.$$

Properties.

- $\overline{(z \pm w)} = \bar{z} \pm \bar{w}$.
- $\overline{(z \cdot w)} = \bar{z} \cdot \bar{w}$.
- $\overline{(\bar{z})} = z$.

Definition. The **absolute value** of a complex number $z = a + b\,\imath$ is given as

$$| z | = \sqrt{a^2 + b^2}.$$

Notice that if we regard a complex number as a vector, the absolute value corresponds to its norm.

Properties.

- $| z | \geq 0$.
- $| z | = 0$ if and only if $Re(z) = Im(z) = 0$.
- $| z \cdot w | = | z | \cdot | w |$.
- $z \cdot \overline{z} = | z |^2 = a^2 + b^2$, where $z = a + b\,\imath$.

Definition. The **multiplicative inverse** of $z = a + b\,\imath$ is given as

$$\frac{1}{z} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\imath.$$

Note. The expression above is obtained when solving for $z \cdot \frac{1}{z} = 1$.

Definition. We define the quotient of z by w as

$$\frac{z}{w} = z \cdot \frac{1}{w}.$$

More precisely, for $z = a + b\,\imath$ and $w = c + d\,\imath$ we have

$$\frac{z}{w} = \left(\frac{ac + bd}{c^2 + d^2}\right) + \left(\frac{cb - ad}{c^2 + d^2}\right)\imath.$$

Properties.

- $\frac{1}{\imath} = -\imath$.
- $\mid \frac{z}{w} \mid = \frac{\mid z \mid}{\mid w \mid}$.
- $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$.

Examples.

$$\frac{1}{(1+3\imath)(2-\imath)} = \frac{1}{5+5\imath} = \frac{5}{50} - \frac{5}{50}\imath = \frac{1}{10} - \frac{1}{10}\imath.$$

$$\frac{1}{1+3\imath} = \frac{1}{10} - \frac{3}{10}\imath.$$

$$\frac{1}{2-\imath} = \frac{2}{5} + \frac{1}{5}\imath.$$

1.2. **Polar Form.** Sometimes it is more convenient to represent a complex number in a different coordinate system. In this section we review the polar form of the complex numbers. One advantage of this representation is that it gives us a geometric description of the product of two complex numbers.

Given a complex number $z = a + b\imath$, the absolute value of z was defined as

$$r = \mid z \mid = \sqrt{a^2 + b^2}.$$

If $z \neq 0$, the angle θ defined by the real axis and the line passing through (a, b) and the origin, will be called the **argument of** Z

$$\arg(z) = \theta.$$

Remark. The argument is not unique,

$$arg(z) = \pm n 2\pi\theta.$$

From trigonometry we know that

$$a = r\cos(\theta), \qquad b = r\sin(\theta).$$

Thus,

$$z = a + b\imath = r\cos(\theta) + r\sin(\theta)\imath = r\left(\cos(\theta) + \imath\sin(\theta)\right).$$

Set

$$e^{\imath\theta} = \cos(\theta) + \imath\sin(\theta),$$

where e is the number known from calculus.

Definition. The **polar form** of $z = a + b\imath$ is

$$z = re^{\imath\theta},$$

where $r = \mid z \mid$ and $\theta = arg(z)$.

Examples.

- If $z = 1 + \sqrt{3}\,\imath$, its polar form is $\quad z = 2\,e^{\imath\,\pi/3}$.
- If $z = 1 - \imath$, its polar form is $\quad z = \sqrt{2}\,e^{\imath\,7\pi/4}$.

Remark.

- For any complex number $z = re^{\imath\theta}$, its inverse multiplicative is

$$z^{-1} = \frac{1}{z} = \frac{1}{r}e^{-\imath\theta}.$$

- $\mid z \mid = 1$ if and only if $z = e^{\imath\theta}$, where θ is any angle.

Examples.

- The polar form of $z = \dfrac{1}{1+\sqrt{3}\,\imath}$ is $\quad z = \dfrac{1}{2e^{\imath\,\pi/3}} = \dfrac{1}{2}\,e^{-\imath\,\pi/3}$.
- The polar form of $z = \dfrac{1}{1-\imath}$ is $\quad z = \dfrac{1}{\sqrt{2}\,e^{\imath\,7\pi/4}} = \dfrac{1}{\sqrt{2}}\,e^{-\imath\,7\pi/4}$.

Theorem 4. If $z = re^{\imath\theta}$ and $w = se^{\imath\eta}$ are two complex numbers, then

$$z \cdot w = \left(re^{\imath\theta}\right)\left(re^{\imath\eta}\right) = rs\, e^{\imath(\theta+\eta)}.$$

Example. For any complex number $z = re^{\imath\theta}$ and any number,

$$z^n = \left(re^{\imath\theta}\right)^n = r^n e^{\imath n\theta}.$$

Example. The polar for of $z = \dfrac{1+\sqrt{3}}{1-\imath}$ is

$$z = \frac{2e^{\imath\pi/3}}{\sqrt{2}\, e^{\imath 7\pi/4}} = \frac{2}{\sqrt{2}} e^{\imath(\pi/3-7\pi/4)} = \frac{2}{\sqrt{2}} e^{-\imath 17\pi/12}.$$

1.3. **Zeroes of Polynomials.** Notice that the complex numbers are a "good" extension of the real numbers in the sense that any algebraic operation in the complex numbers (as $+, -, \cdot, \div$, etc) coincides with the corresponding usual operation when we restrict ourselves to the real numbers.

All of this has been done so that we can talk (finally) of the real use of the complex numbers for us (at least, during this course). We have not only obtained the solution of the equation $x^2 + 1 = 0$, but we can now consider the solution of ANY quadratic equation: given a quadratic equation

$$ax^2 + bx + c = 0,$$

the solutions will be given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The existence of x in the real number is determined by the discriminant $D = b^2 - 4bc$. We have now the following table:

Value of D	Number of Solutions in $\mathbb{R}$	Number of Solutions in $\mathbb{C}$
$D = 0$	unique solution	unique solution
$D > 0$	double solution	double solution
$D < 0$	NO solution	double (pure imaginary) solution

The following example illustrate the procedure.

Example. Find all the solutions of $-3x^2 - 5x - 10 = 0$. We know that

$$x = \frac{5 \pm \sqrt{-95}}{-6}.$$

Now, we have that

$$\sqrt{-95} = \sqrt{-1}\sqrt{95} = \imath\sqrt{95}.$$

Thus,

$$x = -\frac{5}{6} \pm \frac{\sqrt{95}}{6}\imath.$$

Fundamental Theorem of Algebra. We will now mention the most important results about roots of polynomials with coefficients in the complex numbers.

Definition. We say that $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ is a polynomial with complex coefficients if $a_n, a_{n-1}, \ldots, a_1, a0$ are complex numbers. A complex number z is called a **complex root** of $P(x)$ if $P(z) = 0$.

Theorem 1. If z is a root of $P(x)$ then $(x - z)$ is a factor of $P(x)$.

Theorem 2 (The Fundamental Theorem of Algebra). Every non-constant polynomial $P(x)$ with coefficients in the complex numbers has a complex root.

From Theorem 1 and Theorem 2 we conclude that:

(1) If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$, with $a_n \neq 0$, then $P(x)$ has n complex roots.

(2) If $a_n = 1$ then $P(x)$ can be written as

$$P(x) = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n),$$

where $\alpha_1, \ldots, \alpha_n$ are the complex roots of $P(x)$.

Theorem 3. If the complex number z is a root of $P(x)$ then its conjugate $\overline{z}$ is also a root of $P(x)$.

Example. The polynomial $P(x) = x^4 + x^2$ has 4 (complex) roots by Theorem 2. We have that

$$z_1 = 0, \quad z_2 = 0, \quad z_3 = \imath, \quad z_4 = -\imath,$$

are the roots of $P(x)$. Noticed that zero is considered twice.

This can be verified from the factorization as described in Theorem 1:

$$P(x) = x^4 + x^2 = (x - 0)(x - 0)(x - \imath)(x + \imath).$$

Notice that as mentioned in Theorem 3, $\overline{z_3} = z_4$.

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