GEOMETRIC VECTORS

JOSÉ MALAGÓN-LÓPEZ

A **vector** is a physical quantity that is described by both a number (its magnitude) and a direction, and consists of an initial point P_I and a terminal point P_F, and will be denoted as $\vec{P_I P_F}$. We act on such objects by **scalars**, which are physical quantities that can be described by numbers.

1. Vectors in $\mathbb{R}^2$

Let $\mathbb{R}^2$ be the xy-plane. Any point $P = (v_1, v_2)$ in the xy-plane represents a vector $\vec{v} = (v_1, v_2)$ whose initial point is the origin. In other words, $\vec{v} = \vec{OP}$, where O represents the origin.

In this case we say that $\vec{v}$ is in **standard position**, that v_1 and v_2 are the **components** of the vector $\vec{v}$, and that $\vec{v}$ is the **position vector** of the point P.

Note. With this notation, two vectors in standard position are equal if they are equal component-wise.

Sometimes we will use the "transpose" notation $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}^T$.

Note. Given any vector $\vec{v}$ in standard position, a vector with same direction and magnitude than $\vec{v}$, but not neccesarilly same initial and terminal point, is equivalent to $\vec{v}$.

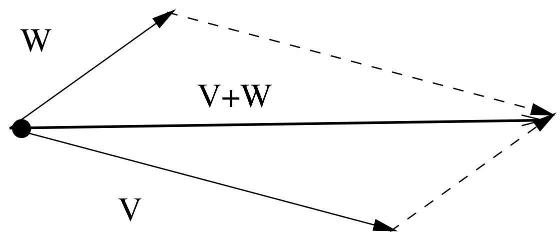
1.1. Basic Operations.

Let a **scalar** will be a real number. The basic arithmetic of vectors is determined by the following:

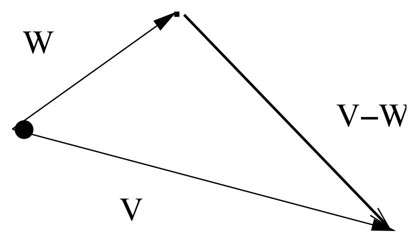
- Addition: $(v_1, v_2) + (w_1, w_2) = (v_1 + w_1, v_2 + w_2)$.
- Subtraction: $(v_1, v_2) - (w_1, w_2) = (v_1 - w_1, v_2 - w_2)$.
- Scalar Multiplication: $\alpha(v_1, v_2) = (\alpha\, v_1, \alpha\, v_2)$, where α is a scalar.

Geometric Interpretation.

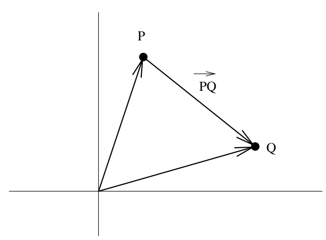
- Vector addition $\vec{v} + \vec{w}$ is given by the "parallelogram law".



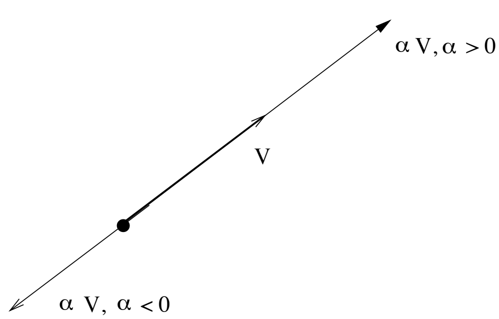
- Vector subtraction $\vec{v} - \vec{w}$ is equivalent to the vector whose initial point is the terminal point of $\vec{w}$ and whose terminal point is the terminal point of $\vec{v}$.



In other words, if $\vec{v} = \vec{OP}$ and $\vec{w} = \vec{OQ}$, then $\vec{v} - \vec{w} = \vec{QP}$.



- Scalar multiplication $\alpha\vec{v}$ is the vector with the same (opposite) direction as $\vec{v}$, but α times as long as $\vec{v}$ if $\alpha > 0$ ($\alpha < 0$, respectively).



Distinguished Vectors. Not all vectors were made equal. We will pay special attention to the following:

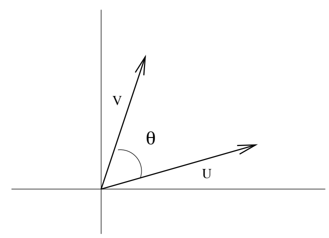
- The Zero Vector: $\vec{0} = (0, 0)$.
- The Standard Unit Vectors: $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$.
- The Negative of a Vector: $-\vec{v} = (-1)\vec{v}$.

1.2. **Special Concepts.** Two vectors $\vec{v}$ and $\vec{w}$ are **parallel** if there is a non-zero scalar α such that $\vec{v} = \alpha\vec{w}$.

Any two vectors in standard position determine an angle θ, where

$$0 \leq \theta \leq \pi.$$

We say that θ is the **angle between $\vec{v}$ and $\vec{w}$**.

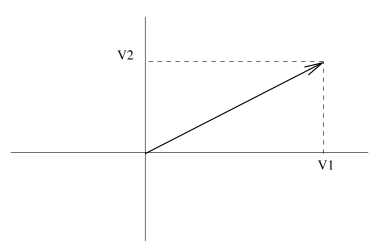


Note. If one of the vectors is zero, then the angle between the vectors is not defined.

Two vectors $\vec{v}$ and $\vec{w}$ are **orthogonal** if the angle between them is a right angle ($\theta = \pi/2$).

1.2.1. *Norm.* For any vector $\vec{v} = (v_1, v_2)$, its **norm** is defined as

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}.$$



With the concept of norm we recover the idea of "magnitude" of a vector.

Properties.

- $\|\vec{v}\| \geq 0$.
- $\|\alpha\,\vec{v}\| = |\,\alpha\,|\ \|\vec{v}\|$.

1.2.2. *Distance.* Given two vectors $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ in standard position, the **distance** between them is defined as

$$d(\vec{v}, \vec{w}) = \|\vec{w} - \vec{v}\| = \sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2}.$$

Property.

- $d(\vec{v}, \vec{w}) = d(\vec{w}, \vec{v})$.

1.2.3. *Dot Product.* The closest notion to a "product" of vectors that we have is the following: the **Dot Product** of the vectors $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ is defined as

$$\vec{v} \bullet \vec{w} = v_1 w_1 + v_2 w_2.$$

Remark. $\vec{v} \bullet \vec{w}$ is always a number, not a vector!

Properties.

- $\vec{v} \bullet \vec{w} = \vec{w} \bullet \vec{v}$.
- $\vec{v} \bullet \vec{0} = 0$.
- $\vec{v} \bullet \vec{v} = \|\vec{v}\|^2$.
- $(\alpha \vec{v}) \bullet \vec{w} = \alpha\,(\vec{v} \bullet \vec{w}) = \vec{v} \bullet (\alpha \vec{w})$.
- $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v}\, +\, \vec{u} \bullet \vec{w}$.

Notice that $\vec{e}_1 \bullet \vec{e}_2 = 0$. This is a particular case of the following result.

Theorem 1. $\vec{v}$ is orthogonal to $\vec{w}$ if and only if $\vec{v} \bullet \vec{w} = 0$.
Proof. $\vec{v}$ being orthogonal to $\vec{w}$ is equivalent to having

$$\|\vec{v} - \vec{w}\| = \|\vec{v} + \vec{w}\|.$$

Computing both sides of the equation using the properties above we conclude that this is equivalent to having

$$-2\vec{v} \bullet \vec{w} = 2\vec{v} \bullet \vec{w}.$$

Which is equivalent to the condition $\vec{v} \bullet \vec{w} = 0$. QED.

1.2.4. *Projection Along a Vector.* Let $\vec{v}$ and $\vec{w}$ be two vectors, $\vec{w} \neq \vec{0}$.

Goal: Write $\vec{v}$ as the sum of two vectors $\vec{v} = \vec{P} + \vec{Q}$, with $\vec{P}$ parallel to $\vec{w}$, and $\vec{Q}$ orthogonal to $\vec{w}$. Notice that:

(1) $\vec{v} = \vec{P} + \vec{Q}$ implies $\vec{Q} = \vec{v} - \vec{P}$.

(2) Condition $\underline{\vec{P} \text{ parallel to } \vec{w}}$ implies that there should be a scalar α such that $\vec{P} = \alpha\vec{w}$.

(3) Condition $\underline{\vec{Q} \text{ orthogonal to } \vec{w}}$ implies that $\vec{Q} \bullet \vec{w} = 0$.

All these considerations imply that

$$0 = \vec{Q} \bullet \vec{w} = \left(\vec{v} - \vec{P}\right) \bullet \vec{w} = (\vec{v} - \alpha\vec{w}) \bullet \vec{w} = (\vec{v} \bullet \vec{w}) - \alpha\,(\vec{w} \bullet \vec{w})\,.$$

Thus, $\alpha = \frac{\vec{v}\bullet\vec{w}}{\vec{w}\bullet\vec{w}}$, and $\vec{P} = \frac{\vec{v}\bullet\vec{w}}{\vec{w}\bullet\vec{w}}\vec{w}$.

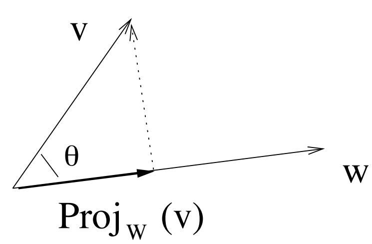
Summarizing: Given two vectors $\vec{v}$ and $\vec{w}$, with $\vec{w} \neq 0$, there is always a vector

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \bullet \vec{w}}{\vec{w} \bullet \vec{w}}\,\vec{w}.$$

called the **projection of $\vec{v}$ along $\vec{w}$** such that

(1) $\text{proj}_{\vec{w}}(\vec{v})$ is parallel to $\vec{w}$.

(2) $\vec{v} - \text{proj}_{\vec{w}}(\vec{v})$ is orthogonal to $\vec{w}$.



The scalar α obtained above is called the **component of $\vec{v}$ along $\vec{w}$**.

Example. The projection of $\vec{v} = (1, 2)$ along $\vec{w} = (-3, -2)$ is the vector

$$\text{proj}_{(-3,-2)}(1, 2) = \frac{(1, 2) \bullet (-3, -2)}{(-3, -2) \bullet (-3, -2)}(-3, -2) = -\frac{7}{13}(-3, -2).$$

1.2.5. *Geometric Interpretation of Dot Product.* Assume $\vec{v} \neq \vec{0}$. Any two vectors $\vec{v}$ and $\vec{w}$ define a unique angle θ, with $0 \leq \theta \leq \pi$, called the **angle between** $\vec{v}$ and $\vec{w}$.

By Law of Cosines we have that

$$\cos(\theta) = \frac{\|\mathrm{proj}_{\vec{w}}(\vec{v})\|}{\|\vec{v}\|} = \left(\frac{\vec{v} \bullet \vec{w}}{\vec{w} \bullet \vec{w}}\right) \frac{\|\vec{w}\|}{\|\vec{v}\|} = \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}.$$

As a consequence we conclude that:

- If $\vec{v} \bullet \vec{w} \leq 0$ then $\pi/2 \leq \theta \leq \pi$.
- If $\vec{v} \bullet \vec{w} \geq 0$ then $\pi/2 \geq \theta \geq 0$.

2. Case $\mathbb{R}^3$

There is no need to stop in $\mathbb{R}^2$. The 3-th space $\mathbb{R}^3$ is the collection of all the ordered triples of the form (v_1, v_2, v_3), where each v_i is a real number, called the *components* of $\vec{v}$.

Similarly, the notions of a vector in standard position and position vector carried out to $\mathbb{R}^3$.

Sometimes we will use the "transpose" notation

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

2.1. **Basic Operations.** Let a **scalar** will be a real number. We define the operations

- Addition: $(v_1, v_2, v_3) + (w_1, w_2, w_3) = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$.
- Subtraction: $(v_1, v_2, v_3) - (w_1, w_2, w_3) = (v_1 - w_1, v_2 - w_2, v_3 - w_3)$.
- Scalar Multiplication: $\alpha(v_1, v_2, v_3) = (\alpha\, v_1, \alpha\, v_2, \alpha\, v_3)$, where α is a scalar.

The geometric interpretation of such operations is the same as in the case of $\mathbb{R}^2$.

Distinguished Vectors.

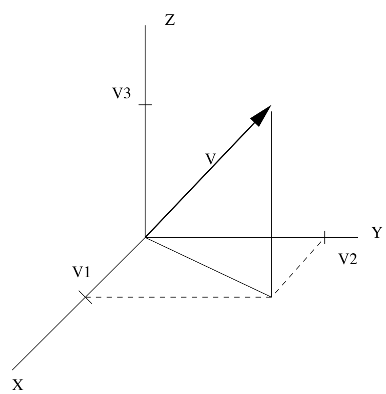
- The Zero Vector: $\vec{0} = (0, 0, 0)$.
- The Standard Unit Vectors: $\vec{e}_1 = (1, 0, 0)$ and $\vec{e}_2 = (0, 1, 0)$, $\vec{e}_3 = (0, 0, 1)$.
- The Negative of a Vector: $-\vec{v} = (-1)\vec{v}$.

2.2. **Special Concepts.** Two vectors $\vec{v}$ and $\vec{w}$ are **parallel** if there is a non-zero scalar α such that $\vec{v} = \alpha\vec{w}$.

Two vectors $\vec{v}$ and $\vec{w}$ are **orthogonal** if the angle between them is a right angle ($\theta = \pi/2$).

2.2.1. *Norm.* For any vector $\vec{v} = (v_1, v_2, v_3)$, its **norm** is defined as

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$



Properties.

- $\|\vec{v}\| \geq 0$.
- $\|\alpha\,\vec{v}\| = |\,\alpha\,|\ \|\vec{v}\|$.

2.2.2. *Distance.* Given two vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$, the **distance** between them is defined as

$$d(\vec{v}, \vec{w}) = \|\vec{w} - \vec{v}\| = \sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2 + (w_3 - v_3)^2}.$$

Property.

- $d(\vec{v}, \vec{w}) = d(\vec{w}, \vec{v})$.

2.2.3. *Dot Product.* The **Dot Product** of the vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ is defined as

$$\vec{v} \bullet \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

Remark. $\vec{v} \bullet \vec{w}$ is always a number, not a vector!

Properties.

- $\vec{v} \bullet \vec{w} = \vec{w} \bullet \vec{v}$.
- $\vec{v} \bullet \vec{0} = 0$.
- $\vec{v} \bullet \vec{v} = \|\vec{v}\|^2$.
- $(\alpha \vec{v}) \bullet \vec{w} = \alpha\,(\vec{v} \bullet \vec{w}) = \vec{v} \bullet (\alpha \vec{w})$.
- $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} \,+\, \vec{u} \bullet \vec{w}$.

Let θ be the angle between two vectors in standard position. We have that

$$\cos(\theta) = \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}.$$

Theorem 1'. $\vec{v}$ is orthogonal to $\vec{w}$ if and only if $\vec{v} \bullet \vec{w} = 0$.

Simillarly, we have again that

- If $\vec{v} \bullet \vec{w} \leq 0$ then $\pi/2 \leq \theta \leq \pi$.
- If $\vec{v} \bullet \vec{w} \geq 0$ then $\pi/2 \geq \theta \geq 0$.

2.2.4. *Projection Along a Vector.* Given two vectors $\vec{v}$ and $\vec{w}$, $\vec{w} \neq 0$, there is always a vector

$$\mathrm{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \bullet \vec{w}}{\vec{w} \bullet \vec{w}}\,\vec{w}.$$

called the **projection of $\vec{v}$ along $\vec{w}$** such that

(1) $\mathrm{proj}_{\vec{w}}(\vec{v})$ is parallel to $\vec{w}$.
(2) $\vec{v} - \mathrm{proj}_{\vec{w}}(\vec{v})$ is orthogonal to $\vec{w}$.

2.3. **Cross Product.** Only for $\mathbb{R}^3$ we can define a product of vectors.

Let $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ be two vectors in $\mathbb{R}^3$. Their **cross product** is the vector defined as

$$\vec{v} \times \vec{w} = (\ v_2 w_3 - v_3 w_2\ ,\ -v_1 w_3 + v_3 w_1\ ,\ v_1 w_2 - v_2 w_1\ ).$$

Remark. The cross product of two vectors in $\mathbb{R}^3$ is a vector in $\mathbb{R}^3$.

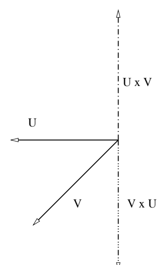
Example: Given $\vec{v} = (2, 4, -1)$ and $\vec{w} = (2, 0, 1)$ we get

$$\begin{aligned}(2,4,-1) \times (2,0,1) &= (4 \cdot 1 - (-1) \cdot 0\, ,\, -(2) \cdot 1 + (-1) \cdot 2\, ,\, 2 \cdot 0 - 4 \cdot 2) \\ &= (4, -4, -8)\end{aligned}$$

and

$$\begin{aligned}(2,0,1) \times (2,4,-1) &= (0 \cdot (-1) - 1 \cdot 4\, ,\, -2 \cdot (-1) + 1 \cdot 2\, ,\, 2 \cdot 4 - 0 \cdot 2) \\ &= (-4, 4, 8).\end{aligned}$$

Theorem 2. If $\vec{v}$ and $\vec{w}$ are non-zero vectors then $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$.



Properties. Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$. Then

- $\vec{v} \times \vec{0} = \vec{0}$.
- $\vec{v} \times \vec{w} = -(\vec{v} \times \vec{w})$.
- $(\alpha \vec{v}) \times \vec{w} = \alpha(\vec{v} \times \vec{w}) = \vec{v} \times (\alpha \vec{w})$.
- $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$.

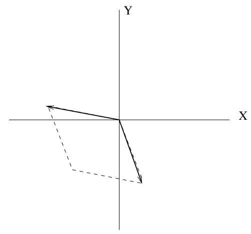
*Remark.*The following two statements are not true in general:

- $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times (\vec{v} \times \vec{w})$.
- If $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$ then $\vec{v} = \vec{w}$.

An important aplication is given in the following theorem.

Theorem 3. Let θ be the angle between two nonzero vectors $\vec{v}$ and $\vec{w}$. Then

(1) $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \cdot \|\vec{w}\| \sin(\theta)$, which is equal to the area of the parallelogram determined by $\vec{v}$ and $\vec{w}$.



(2) In particular, $\vec{v}$ is parallel to $\vec{w}$ if and only if $\vec{v} \times \vec{w} = \vec{0}$.

Example. The area of the triangle whose vertices are the points

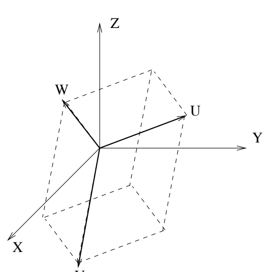
$P = (1, 1, -1)$, $Q = (1, 0, 1)$ and $R = (-1, 1, 0)$ is given by

$$\begin{aligned}\text{Area} &= \frac{\|\vec{PQ} \times \vec{PR}\|}{2} = \frac{\|[(1,0,1) - (1,1,-1)] \times [(-1,1,0) - (1,1,-1)]\|}{2} \\ &= \frac{\|(0,-1,2) \times (-2,0,1)\|}{2} = \frac{\|(-1,4,-2)\|}{2} = \frac{\sqrt{21}}{2}.\end{aligned}$$

One last application.

Theorem 4. The volume of the parallelepiped determined by three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ is given given by

$$\text{Volume} = |\ \vec{u} \bullet (\vec{v} \times \vec{w})\ |\ .$$



Example. The volume of the parallelepiped determined by the vectors $(-1, 2, 0)$, $(1, 0, 4)$ and $(1, -2, 2)$ is equal to

$$\begin{aligned} \text{Volume} &= \mid (-1,2,0) \bullet [(1,0,4) \times (1,-2,2)] \mid \\ &= \mid (-1,2,0) \bullet (8,2,-2) \mid \\ &= \mid -4 \mid = 4. \end{aligned}$$

2.3.1. *Caution!* Do not confuse the formulas

- $\cos(\theta) = \dfrac{\vec{v} \bullet \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}$,

- $\sin(\theta) = \dfrac{\|\vec{v} \times \vec{w}\|}{\|\vec{v}\| \, \|\vec{w}\|}$,

where θ is the angle between $\vec{v}$ and $\vec{w}$.

E-mail address: `jmalagon@uottawa.ca`