NAME: ______________________________

STUDENT NUMBER: ______________________________

- No calculators or other electronic aids allowed.
- No notes, books or other papers allowed.
- Answer all questions in the space provided. You must justify your answers and explain your reasoning.
- There are 6 pages. In all there are 10 questions worth a total of 70 marks.

1. Find $\dfrac{dy}{dx}$.

[3] a) $y = \ln\left(x^2 + y^2\right)$.

Solution: Taking d/dx we get

$$y' = \frac{2x + 2yy'}{x^2 + y^2}.$$

Solving for y' we get

$$y' = \frac{2x}{x^2 + y^2 - 2y}.$$

[3] b) $y = (\sin(x))^{x^2 - e^x}$.

Solution: Taking ln and using properties of logarithms we get

$$\ln(y) = (x^2 - e^x)\ln(sin(x)).$$

Taking d/dx and solving for y' we get

$$y' = \left[(2x - e^x)\ln(\sin(x)) + (x^2 - e^x)\frac{\cos(x)}{\sin(x)}\right](\sin(x))^{x^2 - e^x}.$$

[5] 2. Obtain the linear approximation of $f(x) = \sqrt{1-x}$ at $a = 0$. Then obtain an estimate of $\sqrt{0.9}$.

Solution: Since $f'(x) = \dfrac{-1}{2\sqrt{1-x}}$, we have that the linear approximation of f at $a = 0$ is

$$L_0(x) = f'(0)(x - 0) + f(0) = -\frac{x}{2} + 1.$$

Since $\sqrt{0.9} = \sqrt{1-x}$ when $x = 0.1$, we conclude

$$\sqrt{0.1} \approx L_0(0.1) = -\frac{0.1}{2} + 1 = \frac{19}{20} = 0.95$$

[5] 3. Assume that the height of a triangle grows at a rate of 1 cm/sec, while its area increases at a rate of 2 cm^2/sec. Find the rate at which the length of the base is changing when the height is 10 cm and the area is 100 cm^2.

Solution: Let A, h and b be the area, height and base of the triangle, respectively. Then $A = \frac{bh}{2}$.

The data means that $A' = 2$ and $h' = 1$.

Also, when $A = 100$ and $h = 10$, we conclude from the area formula that $b = 20$.

Now, taking d/dt on the area formula we get

$$A' = \frac{1}{2}\left(b' \cdot h + b \cdot h'\right).$$

Plugging in values and solving for b' we get

$$b' = -\frac{16}{10} = -1.6$$

[6] 4. Suppose that you have 8 meters of wire to be use to built a square and a circle. How much of the wire should be used for the square and how much should be used for the circle in order to enclose the maximum total area?

Solution: Let x be the side of the square. Let r be the radius of the circle. Then the total area is

$$T = x^2 + \pi r^2.$$

Since we have 8 meters of wire to use we have the equation

$$8 = 4x + 2\pi r$$

Solving for r we get $r = \frac{4-2x}{\pi}$. Writing T as a function on x we obtain

$$T(x) = \frac{(4+\pi)x^2 - 16x + 16}{\pi}, \qquad 0 \geq x \geq 2.$$

Since the graph of T is a parabola open upward, we have that the maximum will be reached on the endpoints.

Since

$$T(0) = \frac{16}{\pi}, \qquad > \qquad T(2) = 4,$$

we conclude that the maximum area is obtained when we use all the wire on the circle.

5. Consider

$$f(x) = \frac{x^2 + 7x + 6}{x^2}, \qquad f'(x) = \frac{-7x - 12}{x^3}, \qquad f''(x) = \frac{14x + 36}{x^4}.$$

[3] a) Find the domain, x-intercepts and y-intercept of f.

Solution:

The domain of f consists of all the non-zero real numbers.

Since $x = 0$ is not in the domain, f has no y-intercept.

The x intercepts are given by the solution of $0 = x^2 + 7x + 6 = (x+6)(x+1)$. Thus, the x-intercepts are $x = -6, -1$.

[3] b) Find the horizontal and vertical asymptotes of f.

Solution:

Since

$$\lim_{x\to\infty} \frac{x^2 + 7x + 6}{x^2} = 1 = \lim_{x\to-\infty} \frac{x^2 + 7x + 6}{x^2},$$

we conclude that f has an horizontal asymptote at $y = 1$.

Since

$$\lim_{x\to 0} \frac{x^2 + 7x + 6}{x^2} = \lim_{x\to 0} \frac{x^2}{x^2}\left(\frac{1 + 7/x + 6/x^2}{1}\right) = \lim_{x\to 0} 1 + \frac{7}{x} + \frac{6}{x^2} = \infty$$

we conclude that f has a vertical asymptote at $x = 0$.

[5] c) Determine the intervals where f is increasing and where is decreasing. Give the maximum and minimums of f.

Solution:

Since $0 = f'(x)$ when $0 = -7x - 12$, we have that $x = -\frac{12}{7}$ is a candidate for maximum or minimum.

Since the intervals where we need to verify where f is increasing/decreasing by the numbers where f' is not defined and where f' is equal to zero, we have

Interval	f'	f
$(-\infty, -12/7)$	$f' = \frac{+}{-}$	f decreasing
$(-12/7, 0)$	$f' = \frac{-}{-}$	f increasing
$(0, \infty)$	$f' = \frac{-}{+}$	f decreasing

Hence, f has a minimum at $x = -12/7$.

[5] d) Determine the intervals where f is concave upward and where is concave downward. Give the inflections points of f.

Solution:

Since $0 = f''(x)$ when $0 = 14x + 36$, we have that $x = -\frac{18}{7}$ is a of inflection point.

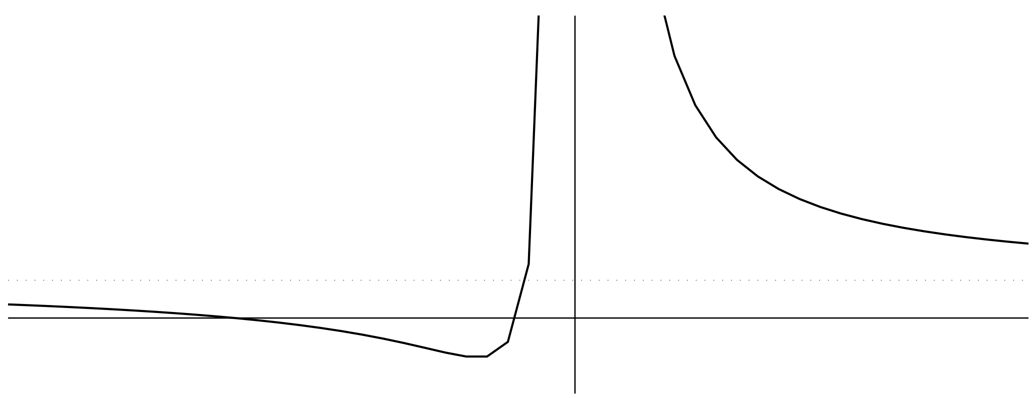
Since the intervals where we need to verify where f is concave up/concave down by the numbers where f'' is not defined and where f'' is equal to zero, we have

Interval	f''	f
$(-\infty, -18/7)$	$f'' = \frac{-}{+}$	f concave down
$(-18/7, 0)$	$f'' = \frac{+}{+}$	f concave up
$(0, \infty)$	$f'' = \frac{+}{+}$	f concave up

Hence, f has an inflection point at $x = -18/7$.

[3] e) Sketch the graph of $f(x)$.

Solution:



[6] 6. Find $\lim\limits_{x\to 0^+} (1+\sin(4x))^{\frac{1}{x}}$.

Solution: We have that

$$\lim_{x\to 0^+} (1+\sin(4x))^{\frac{1}{x}} = \lim_{x\to 0^+} e^{\frac{\ln(1+\sin(4x))}{x}} = \lim_{u\to 4} e^u = e^4$$

since

$$\lim_{x\to 0^+} \frac{(1+\sin(4x))}{x} \overset{0/0}{=} \lim_{x\to 0^+} \frac{\left(\frac{4\cos(4x)}{1+\sin(4x)}\right)}{1} = 4.$$

7. Consider the function $f(x) = 1 + x^2$.

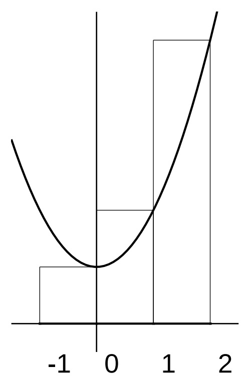
[4] a) Use the right endpoints as sample points and obtain an approximation of $\int_{-1}^{2} f(x)\,dx$ by using three rectangles.

Solution: Set $\Delta_x = \frac{2-(-1)}{3} = 1$, so $x_0 = -1, x_1 = 0, x_2 = 1, x_3 = 2$. Thus

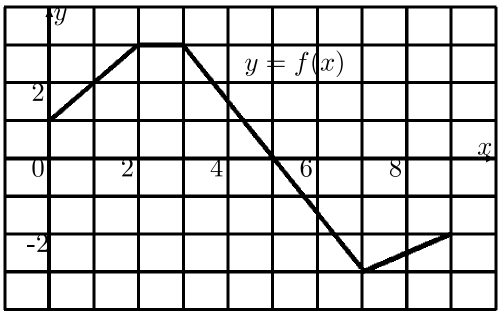
$$\int_{-1}^{2} (1+x^2)dx \approx 1 + 2 + 5 = 8.$$

[2] b) Draw the curve $y = f(x)$ and the rectangles used in part a).

Solution:



[5] 8. The following is the graph of $f(x)$. Find $\int_0^9 f(x)\,dx$.



Solution: $\int_0^9 f(x)\,dx = 10 - 8 = 2.$

9. Let f and g be functions such that $\int_{-1}^3 f(x)\,dx = 10$, $\int_2^3 f(x)\,dx = 3$, and $\int_{-1}^3 g(x)\,dx = 2$.

[3] a) Find $\displaystyle\int_{-1}^2 f(x)\,dx$.

Solution: $\displaystyle\int_{-1}^2 f(x)\,dx = \int_{-1}^3 f(x)\,dx - \int_2^3 f(x)\,dx = 10 - 3 = 7.$

[3] b) Find $\displaystyle\int_{-1}^3 \left(\frac{f(x)}{4} - 2g(x)\right)\,dx$.

Solution: $\displaystyle\int_{-1}^3 \left(\frac{f(x)}{4} - 2g(x)\right)\,dx = \frac{1}{4}\int_{-1}^3 f(x)\,dx - 2\int_{-1}^3 g(x)\,dx = \frac{10}{4} - 4 = -\frac{3}{2}.$

[6] 10. If $f(x)$ is continuous, $f'(3) = 5$ and $f(3) = 0$, find

$$\lim_{x\to 0} \frac{f(3-x) + f(2x+3)}{x}.$$

Solution: $\lim_{x\to 0} \frac{f(3-x)+f(2x+3)}{x} \overset{0/0}{=} \lim_{x\to 0} \frac{-f'(3-x)+2f'(2x+3)}{1} = -5 + 10 = 5.$