MATH 1009 - A TEST 4 - SOLUTIONS

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1. [**5 Marks**] Find the absolute maximum value and the absolute minimum value of the function $f(x) = \dfrac{5}{x^2+1}$ on the interval $[-1, 2]$.

Solution:

The domain of f is the set of all real numbers.

$f'(x) = (5(x^2+1)^{-1})' = -5(x^2+1)^{-2}(x^2+1)' = -\dfrac{10x}{(x^2+1)^2} = 0$ when $x = 0$. Since the derivative is defined for all real x (the denominator is never zero), the function has one critical number $x = 0$, which belongs to the interval $[-1, 2]$.

It remains to compare the value of f at the endpoints of the interval $[-1, 2]$ and at the critical number:

$f(-1) = 5/2$, $f(2) = 1$, $f(0) = 5$. Thus, the absolute maximum value is $f(0) = 5$ and the absolute minimum value is $f(2) = 1$.

2. [**4 Marks**] Find all the inflection points of the function $f(x) = x^3 - 3x^2 + x + 1$. (Give both the x- and the y-coordinates.)

Solution:

The domain of f is the set of all real numbers.

$f'(x) = 3x^2 - 6x + 1, / f''(x) = 6x - 6 = 0$ when $x = 1$. Since f'' is changing its sign as the graph is traversed from left to right across $x = 1$, then $(1, f(1)) = (1, 0)$ is an inflection point of the graph of f.

3. [**19 Marks**] Consider the function $f(x) = \dfrac{x}{x-2}$.

[1] **(a)** Find the domain of f.

[3] **(b)** Find the horizontal and the vertical asymptotes of the graph of f, if any.

[6] **(c)** Determine the intervals where f is increasing and where f is decreasing. Find the relative extrema, if any.

[4] **(d)** Determine the intervals where f is concave up and where f is concave down.

[1] **(e)** Find the inflection points of f, if any.

[2] **(f)** Show that there is an x-intercept in the interval $[-2, 1]$. (Hint: use the Intermediate Value Theorem).

[2] **(g)** Sketch the graph of f. Mark all the points you found in parts (a) - (f).

Solution:

(a) $Domain\{f\} = (-\infty, 2) \bigcup (2, \infty)$.

(b) $\lim_{x\to\infty} f(x) = 1 = \lim_{x\to-\infty} f(x)$, so $y = 1$ is a horizontal asymptote.

$\lim_{x\to 2+} f(x) = \infty$, $\lim_{x\to 2-} f(x) = -\infty$, so $x = 2$ is a vertical asymptote.

(c) $f'(x) = \dfrac{1\cdot(x-2) - x\cdot 1}{(x-2)^2} = -\dfrac{2}{(x-2)^2}$. Since $f' < 0$ for all x in the domain, the function is decreasing for all such x. Since $f'(x)$ is never 0, there are no relative extrema.

(d) $f'' = \dfrac{4(x-2)}{(x-2)^4} = \dfrac{4}{(x-2)^3}$. $f'' > 0$ when $x > 2$, so the graph is concave up (CU) for $x \in (2, \infty)$. $f'' < 0$ when $x < 2$, so the graph is concave down (CD) for $x \in (-\infty, 2)$.

(e) $f'' \neq 0$, therefore the graph of f has no inflection points.

(f) $f(-2) = \dfrac{1}{2} > 0$, $f(1) = -1 < 0$. Since the values of the function have the opposite sign at the endpoints of the interval, then there must be at least one number x in the interval where $f(x) = 0$.

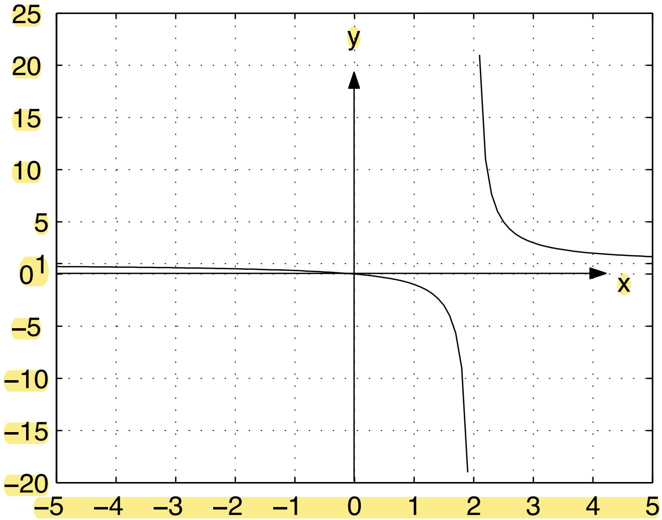

Figure 1: $f(x) = \dfrac{x}{x-2}$

4. [**12 Marks**] If exactly 80 people buy a yoga club membership, the price is \$160 per person per year. If more people join, then the price is reduced by \$1 for each additional person. Determine how many members will result in a maximum revenue for the club. Find the optimal price of membership.

Solution:

Let x be the number of additional members. Then the total number of club members is $(80 + x)$, and the new price for the membership is

$$p(x) = 160 - 1\cdot x = 160 - x \quad \text{(dollars)}.$$

Thus, the revenue of the club to be maximized is given by the product of the number of members and the price of membership, that is,

$$R(x) = (80 + x)(160 - x) = 12,800 + 80x - x^2.$$

Since the price $p(x) = 160 - x$ must be a nonnegative number, the restriction $160 - x \geq 0$ requires $x \leq 160$. Also, as x denotes the number of additional members of the club, $x \geq 0$. Thus, we have to maximize $P(x)$ over the domain $0 \leq x \leq 160$.

Compute the first derivative of $R(x)$ and find all the critical numbers in the domain.

$$R'(x) = 80 - 2x = 0 \quad \text{when} \quad 2x = 80, \quad \Rightarrow \quad x = 40.$$

The critical number $x = 40$ is in the domain. Since we have to find the absolute maximum of $R(x)$ on the closed interval $[0, 160]$, we compute

$$R(0) = 12,800; \quad R(40) = 14,400; \quad R(160) = 0.$$

Thus, the largest value is $R(40) = 12,800$, so the maximum revenue is obtained when the club has 40 extra members signed up. Then the optimal price of membership is $p = 80 + 40 = 120$ dollars.