Questions 1–6 are multiple choice. Enter the **letter** corresponding to each correct answer in the appropriate box below.

Question	1	2	3	~~4~~	5	6
Answer	C	C	E		D	B

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1. The truth table of a compound proposition p with atomic propositions A, B, and C is as follows:

A	B	C	p
T	T	T	T ←
T	T	F	F
T	F	T	F
T	F	F	T ←
F	T	T	F
F	T	F	T ←
F	F	T	F
F	F	F	F

Only one of the following propositions is a **disjunctive normal form** of p — which one?

A. $A \vee B \vee C$
B. $(A \vee B \vee C) \wedge (A \vee \neg B \vee \neg C)$
C. $(A \wedge B \wedge C) \vee (A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C)$
D. $(A \vee B \vee C) \wedge (A \vee \neg B \vee \neg C) \wedge (\neg A \vee B \vee \neg C)$
E. $A \wedge B \wedge C$
F. $(A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C)$

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2. Let $A = \{\emptyset, 1, \{1\}, 2, \{1,3\}\}$. What is the **cardinality of the power set of** A?

A. 8 **B.** 16 **C.** 32 **D.** 64 **E.** 128 **F.** None of the above.

$|A| = 5$

$|\mathcal{P}(A)| = 2^{|A|} = 2^5 = 32$

3. Which of the following statements are **true**?

   $F \quad \overbrace{F \vee T}^{T}$

   (i) If X is false, Y is false, and Z is true, then $X \rightarrow (Y \vee Z)$ is true. ✓

   (ii) The compound propositions $\neg(p \rightarrow (q \rightarrow \neg r))$ and $p \wedge q \wedge r$ are logically equivalent. ✓

   (iii) The compound proposition $p \rightarrow (\neg q \rightarrow \neg p)$ is a tautology. X

   (iv) If the set of premises of an argument is consistent, then the argument is valid. X

   A. only (iii) **B.** only (iv) **C.** only (i) **D.** (i) and (iii)

   E. (i) and (ii) **F.** only (ii)

(ii) ($p \wedge q \wedge r$ is T) iff (p, q, r all T)

($\neg(p \Rightarrow (q \Rightarrow \neg r))$ is T) iff (p is T and $q \Rightarrow \neg r$ is F)

iff (p is T, q is T, r is T)

(iii) ($p \Rightarrow (\neg q \Rightarrow \neg p)$ is F) iff (p is T, and $\neg q \Rightarrow \neg p$ is F)

iff (p is T, q is F, p is T)

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4. On the Island of Knights and Knaves you meet two inhabitants A and B. Person A says: "I am a knave only if B is a knave." Only one of the following statements is **true**. Which one?

   (i) A is a knave and B is a knight.

   (ii) A is a knight and B is a knave.

   (iii) A and B are both knights.

   (iv) A and B are both knaves.

   (v) A is a knight but it is impossible to determine what B is.

   (vi) B is a knight but it is impossible to determine what A is.

   A. (i) **B.** (ii) **C.** (v) **D.** (vi) **E.** (iv) **F.** (iii) G. none of the above

p: "A is a knight."

q: "B is a knight."

A says: $\neg p \rightarrow \neg q$

p	q	$\neg p \rightarrow \neg q \equiv q \rightarrow p$	
T	T	T	←
T	F	T	←
F	T	F	←
F	F	T	

5. Let $S = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. Only one of the following statements is **false**. Which one?

   (i) $\emptyset \subseteq S$ ✓
   (ii) $\emptyset \in S$ ✓
   (iii) $\{\emptyset\} \subseteq S$ ✓
   (iv) $\{\emptyset, \{\emptyset\}\} \in S$ ✓
   (v) $\{\emptyset, \{\emptyset\}\} \subseteq S$ ✓
   (vi) $\{\{\emptyset\}\} \in S$ X

   A. (iii) **B.** (iv) **C.** (v) **D.** (vi) **E.** (ii) **F.** (i)

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6. Let A and B be finite sets, and $f : A \to B$ and $g : B \to A$ two functions. Which of the following statements are **true**?

   (i) If $g \circ f$ is one-to-one, then f is one-to-one. ✓
   (ii) If $g \circ f$ is onto, then f is onto. X
   (iii) If $|A| < |B|$, then f can not be onto. ✓
   (iv) If f is a bijection, then g is also a bijection. X
   (v) If $|A| < |B|$, then f can not be one-to-one. X

   A. (i) and (ii) **B.** (i) and (iii) **C.** (ii) and (iv) **D.** (ii) and (v) **E.** only (i)

7. Let A, B, and C be subsets of the universal set U. Use properties of set operations and set identities to show the following. *You need not name the identities used.*

$$A - (\overline{B} - C) = (A \cap B) \cup (A \cap C)$$

$$\begin{aligned}
A - (\overline{B} - C) &= A \cap \overline{(\overline{B} - C)} \\
&= A \cap \overline{(\overline{B} \cap \overline{C})} \\
&= A \cap (\overline{\overline{B}} \cup \overline{\overline{C}}) \\
&= A \cap (B \cup C) \\
&= (A \cap B) \cup (A \cap C)
\end{aligned}$$

8. Define the following atomic propositions:

   S: "Parking regulations are very strict."
   N: "Parking regulations allow no exceptions."
   Q: "Parisians question parking regulations."
   M: "Michel is the mayor of Paris."
   T: "Traffic in Paris is improved."

   Translate each of the following sentences into compound logical propositions using the atomic propositions S, N, Q, M, and T as defined above.

   (a) Traffic in Paris will be improved only if Parisians do not question parking regulations.

   $T \to \neg Q$

   (b) For parking regulations to be very strict, it is sufficient that Michel be the mayor of Paris.

   $M \to S$

   (c) For traffic in Paris to improve, it is necessary that parking regulations allow no exceptions and that Michel be the mayor of Paris.

   $T \to (N \wedge M)$

   (d) If parking regulations are very strict but allow no exceptions, then Parisians do not question them.

   $(S \wedge N) \to \neg Q$

9. Use any method you know to determine whether or not the argument below is valid. If the argument is not valid, give a counterexample.

$Q \leftrightarrow R$
$\neg(R \vee \neg P)$
$\neg\,(P \leftrightarrow (R \rightarrow P))$

$\therefore \neg(Q \rightarrow P)$

$Q \Leftrightarrow R$ ✓
$\neg(R \vee \neg P) \equiv \neg R \wedge P$ ✓
$\neg(P \Leftrightarrow (R \Rightarrow P))$ ✓
$Q \Rightarrow P \equiv \neg Q \vee P$ ✓

Q	$\neg Q$
R	$\neg R$
$\neg R$	$\neg R$
P	P
X	

Under the right branch:

P	$\neg P$
$R \wedge \neg P \equiv \neg(R \Rightarrow P)$ ✓	$R \Rightarrow P$
	X

Under P:

$\neg Q$	P
R	R
$\neg P$	$\neg P$
X	X

The argument is valid.

P	Q	R	$Q \Leftrightarrow R$	$\neg(R \vee \neg P)$ $\equiv \neg R \wedge P$	$\neg(P \Leftrightarrow (R \Rightarrow P))$ $\equiv \neg P \wedge \neg R$	$\neg(Q \Rightarrow P)$ $\equiv Q \wedge \neg P$
T	T	T	T			
T	T	F		T		
T	F	T				
T	F	F	T	T		
F	T	T	T			T
F	T	F			T	T
F	F	T				
F	F	F	T		T	

The 3 premises are never T at the same time.

Hence, the conclusion is T whenever all premises are T (vacuously)

The argument is valid.

10. Give an **indirect proof** of the following theorem.

$\underbrace{\textit{If } n^2 - 3n \textit{ is an odd integer,}}_{p}$ *then* $\underbrace{n \textit{ is an even integer.}}_{q}$

We must prove $p \to q$ using an indirect proof.

Here, p: "$n^2 - 3n$ is odd."
q: "n is even."

Assume $\neg q$: "n is odd". Hence $n = 2k+1$ for $k \in \mathbb{Z}$. ①

Then:

$$n^2 - 3n = (2k+1)^2 - 3(2k+1) = 4k^2 + 4k + 1 - 6k - 3$$
$$= 2(2k^2 - k - 1) = 2m$$ ①

Since $m = 2k^2 - k - 1 \in \mathbb{Z}$, $n^2 - 3n$ is even. ①

Thus we showed $\neg q \to \neg p$.

① set-up & conclusion

Therefore $p \to q$ is T.

11. Prove that the function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by

$$F(x, y) = (y, x + y)$$

is a bijection.

We must show F is one-to-one and onto.

To show F is one-to-one:

Take any $(x_1, y_1), (x_2, y_2) \in \mathbb{R} \times \mathbb{R}$.

Suppose $F(x_1, y_1) = F(x_2, y_2)$,

i.e. $(y_1, x_1 + y_1) = (y_2, x_2 + y_2)$.

Then $\begin{cases} y_1 = y_2 \\ x_1 + y_1 = x_2 + y_2 \end{cases}$

Substituting $y_1 = y_2$ into the 2nd eq. yields $x_1 = x_2$.

Thus $F(x_1, y_1) = F(x_2, y_2)$ implies $(x_1, y_1) = (x_2, y_2)$, and so F is one-to-one.

To show F is onto:

Take any $(a, b) \in \mathbb{R} \times \mathbb{R}$ (codomain). We must find $(x, y) \in \mathbb{R} \times \mathbb{R}$ (domain) s.t. $F(x, y) = (a, b)$,

i.e. $(y, x + y) = (a, b)$.

Solving $\left\{ \begin{array}{l} y = a \\ x + y = b \end{array} \right\}$, we obtain $x = b - a$, $y = a$.

Thus $F(b - a, a) = (a, b)$ for all $(a, b) \in \mathbb{R} \times \mathbb{R}$, showing that F is onto.

Since F is one-to-one and onto, it is a bijection.