UNIVERSITÉ D'OTTAWA • UNIVERSITY OF OTTAWA

Sciences • Science

Mathématiques et statistique
Mathematics and Statistics

MAT 1341A Final Exam

December 6, 2002 Duration: 3 hours.

Instructor: Barry Jessup.

Family Name:______________________

First Name:________________________

Student number:____________________

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PLEASE READ THESE INSTRUCTIONS CAREFULLY.

1. You have 3 hours to complete this exam.
2. This is a closed book exam, and no notes of any kind are allowed. **The use of calculators**, cell phones, pagers or any text storage or communication device **is not permitted.**
3. Read each question carefully -you will save yourself time and unnecessary grief later on.
4. Questions 1 to 10 are multiple choice. These questions are worth 2 points each and no part marks will be given. Please record your answers in the space provided above.
5. Questions 11 – 15 require a complete solution, and are worth 6 points each, so spend your time accordingly.
6. **The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct.**
7. You must answer these questions in the space provided. Use the backs of pages if necessary.
8. Where it is possible to check your work, do so.
9. Good luck! Bonne chance!

1. Which of the following are bases for $\mathbf{R}^3$?
(1) $\{(4,\ 2,\ 0),\ (1,\ 4,\ 1),\ (5,\ 6,\ 1)\}$
(2) $\{(-1,\ 2,\ 3),\ (3,\ 3,\ 2)\}$
(3) $\{(-1,\ 3,\ -5),\ (1,\ -2,\ 4),\ (2,\ 0,\ 4),\ (5,\ 1,\ 9)\}$

A. All three
B. (1) only
C. (2) only
D. (1) and (2)
E. (2) and (3)
F. None of them

2. The dimension of $S = \{A \in \mathbf{M}_{22}(\mathbf{R}) \mid \text{trace } A = 0\}$ is:

A. 0
B. 1
C. 2
D. 3
E. 4
F. S is not a vector space.

3. Consider a homogeneous system of 6 linear equations in 5 unknowns. Which of the following is true?

A. The system always has infinitely many solutions.
B. The system has only the trivial solution.
C. The system has either the trivial solution only or infinitely many solutions.
D. The system can have no solution.
E. The system has between 1 and 6 solutions.
F. The system has between 0 and 5 solutions.

4. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, find the main diagonal of A^{-1}.

A. (0, 1, 0)
B. (-1, 0, -1)
C. (-1, -1, 1)
D. (0, -1, 0)
E. (-1, 0, 0)
F. (0, 0, 1)

5. The dimension of the subspace of $\mathbf{R}^5$ spanned by the vectors $(1,\ 0,\ 3,\ 1,\ 1)$, $(1,\ -1,\ 7,\ -1,\ 0)$ $(2,\ 1,\ 2,\ 4,\ 3)$ and $(5,\ 1,\ 11,\ 7,\ 6)$ is:

A. 5
B. 4
C. 3
D. 2
E. 1
F. 0

6. Which of the following are linearly independent in $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$.?

$S = \{\cos x, \sin x\}$
$T = \{1, \cos^2 x, \sin^2 x\}$
$U = \{1, 2\cos^2 x, 3\sin^2 x\}$
$V = \{2\cos x, 3\sin x\}$

A. T and V.
B. S, U and T.
C. T and U.
D. S and T.
E. S and V.
F. S, U and V.

7. Compute $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{2002}$.

A. $\begin{bmatrix} 1 & 0 & -6006 & 0 \\ 0 & 1 & 0 & 6006 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 & 6006 \\ 0 & 1 & -6006 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 & -3^{2002} \\ 0 & 1 & 3^{2002} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 & 3^{2002} \\ 0 & 1 & -3^{2002} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

F. $\begin{bmatrix} 1 & 0 & 0 & -6006 \\ 0 & 1 & 6006 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

8. Compute the determinant $\begin{vmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{vmatrix}$.

A. -10
B. 18
C. 30
D. 15
E. -18
F. -36

9. Let A be a 4×5 matrix such that $\text{rank}(A) = 4$. If $\ker A = \{x \in \mathbf{R}^5 \mid Ax = 0\}$ and $\text{col } A = \{Ax \mid x \in \mathbf{R}^5\}$, then

A. $\text{col } A = \mathbf{R}^4, \dim \ker A = 4$
B. $\dim \text{col } A = 4, \ker A = \mathbf{R}^5$
C. $\text{col } A = \{0\}, \dim \ker A = 4$
D. $\dim \text{col } A = 1, \dim \ker A = 1$
E. $\dim \text{col } A = 1, \ker A = \mathbf{R}^5$
AF. $\text{col } A = \mathbf{R}^4, \dim \ker A = 1$

10. Let $A = \begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 1 & 3 & 0 \\ 2 & 1 & 3 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}$. If $x \in \mathbf{R}^4$, the dimension of the solution-space of $Ax = 0$ is:

A. Infinite
B. 4
C. 3
D. 2
E. 1
F. 0

11. Consider the linear system

$$\begin{array}{rcrcrcl} x & + & 3y & + & z & = & 1 \\ -x & - & 2y & + & pz & = & 1 \\ 3x & + & 7y & - & z & = & q \end{array}$$

a) If $[A \,|\, b]$ is the augmented matrix of the system above, find $\operatorname{rank} A$ and $\operatorname{rank}[A \,|\, b]$ for all values of p and q.

b) Find all p and q so that this system has
   i) a unique solution,
   ii) infinitely many solutions, and
   iii) no solutions.

c) In case (ii) above, give a complete geometric description of the set of solutions.

12. Suppose $v_1 = (1, -1, 0, 1)$, $v_2 = (1, -1, 1, 0)$, $v_3 = (2, -2, -1, 3)$, and $v_4 = (-1, 1, 2, -3)$ and let $W = \mathrm{span}\{v_1, v_2, v_3, v_4\}$.

a) Find any basis of W, and so determine $\dim W$.

b) Find a basis of W which is a subset of $\{v_1, v_2, v_3, v_4\}$.

c) Extend your basis in (b) (if necessary) to a basis of $\mathbf{R}^4$.

13. Let $W = \{(x, y, z) \in \mathbf{R}^3 \mid x - y + z = 0\}$.

a) Find a basis of W and give the dimension of W.

b) Find an orthogonal basis of W.

c) Find the projection of $(1, 0, 0)$ onto W.

14. Let $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$.

a) Check that -3 and 1 are the eigenvalues of A.

b) Find a basis of $E_1 = \{x \in \mathbf{R}^3 \mid Ax = 1x\}$.

c) Find a basis of $E_1{-3} = \{x \in \mathbf{R}^3 \mid Ax = -3x\}$.

d) Find an invertible matrix P such that $P^{-1}AP = D$ is diagonal, and give this diagonal matrix D. Show why our choice of P is invertible

15. a) Let A be a real $n \times n$ matrix. Give 4 statements equivalent to "$\operatorname{rank} A = n$", in terms of

(I) the columns of A

(II) the determinant of A

(III) the homogeneous system $Ax = 0$

(IV) the row canonical form of A

15b) State whether the following are true or false. If true, explain why, if false, give a numerical example to illustrate.

i) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable.

ii) The dimension of $\{v \in \mathbf{R}^5 \mid v \cdot (1,2,3,4,5) = 0\}$ is 4.

iii) $\{v_1, v_2, v_3\}$ linearly independent implies that $\{v_1, v_1+v_2, v_1+v_2+v_3\}$ is linearly independent.

ii) If A is an 11×11 matrix, and $A^t = -A$, then $\det A = 0$.