# TEST 1 SOLUTIONS–MATH 1102

/10 1. Row-reduce the augmented matrix of the linear system in 3 unknowns

$$\begin{aligned} 2x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 + 3x_3 &= 3 \\ -x_1 + x_3 &= 1 \end{aligned}$$

over $\mathbb{R}$ into RREF. Indicate every row operation clearly.

**Solution:** The augmented matrix of this system is

$$\begin{bmatrix} 0 & 2 & 2 & 0 \\ 3 & 3 & 3 & 3 \\ -1 & 0 & 1 & 1 \end{bmatrix}$$

We row-reduce as follows.

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 3 & 3 & 3 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{(1/3)R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{(1/2)R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

**Grading:** 2 for augmented matrix, 2 for correct RREF, 6 for computation.

/10 2. Row-reduce the matrix

$$\begin{bmatrix} -\dot{3} & \dot{2} \\ \dot{4} & \dot{1} \end{bmatrix} \in M_{22}(\mathbb{Z}_7)$$

over the field $\mathbb{Z}_7$ into RREF. Write your final answer *without* any additive inverse notation (no minus signs). Indicate every row operation clearly.

**Solution:** $-\dot{3} = \dot{4}$ in $\mathbb{Z}_7$, since $\dot{4}+\dot{3} = ((4+3\dot{)}\mathrm{mod}\,7) = \dot{0}$. Therefore, the above matrix is equal to

$$\begin{aligned} \begin{bmatrix} \dot{4} & \dot{2} \\ \dot{4} & \dot{1} \end{bmatrix} & \xrightarrow{R_2 - R_1} \begin{bmatrix} \dot{4} & \dot{2} \\ \dot{0} & -\dot{1} \end{bmatrix} \\ & \xrightarrow{\dot{4}^{-1}R_1} \begin{bmatrix} \dot{1} & \dot{4}^{-1}\dot{2} \\ \dot{0} & -\dot{1} \end{bmatrix} \\ & \xrightarrow{-\dot{1}R_2} \begin{bmatrix} \dot{1} & \dot{4}^{-1}\dot{2} \\ \dot{0} & \dot{1} \end{bmatrix} \end{aligned}$$

We have used $(-\dot{1})^2 = \dot{1}$ in the previous step. Now, we could compute $\dot{4}^{-1} = \dot{2}$ and continue with explicit computations or we could be clever and lazy and continue with

$$\xrightarrow{R_1 + (-\dot{4}^{-1}\dot{2})R_2} \begin{bmatrix} \dot{1} & \dot{0} \\ \dot{0} & \dot{1} \end{bmatrix}$$

**Grading:** 3 for removing additive inverse notation correctly. 4 for remaining modular arithmetic. 3 for row-reduction.

3. Suppose $F$ is a field. Using only the field axioms and $a0 = 0$, prove

$$(-1)a = -a$$

/10 for all $a \in F$. Justify each step in your proof. (Hint: distributivity.)

**Solution:**

$$\begin{aligned} & -1 + 1 = 0 \ \ \text{additive inverse} \\ \Rightarrow \ & a(-1+1) = a0 \ \ \text{multiply both side by } a \\ \Rightarrow \ & a(-1+1) = 0 \ \ \text{given} \\ \Rightarrow \ & a(-1) + a1 = 0 \ \ \text{distributivity} \\ \Rightarrow \ & (-1)a + a1 = 0 \ \ \text{commutativity} \\ \Rightarrow \ & (-1)a + a = 0 \ \ \text{multiplicative identity} \\ \Rightarrow \ & ((-1)a + a) + (-a) = 0 + (-a) \ \ \text{add additive inverse of } a \\ \Rightarrow \ & (-1)a + (a + (-a)) = 0 + (-a) \ \ \text{associativity of addition} \\ \Rightarrow \ & (-1)a + 0 = 0 + (-a) \ \ \text{additive inverse} \\ \Rightarrow \ & (-1)a = -a \ \ \text{additive identity} \end{aligned}$$

**Grading:** 4 for justification, 2 for distributivity, 4 for correct argument.

/3 4. Suppose $a$ is an integer and $n$ is an integer greater than 1. Provide the definition of $a \bmod n$.

**Solution:** $a \bmod n = r$ where $r$ is the remainder with $0 \leq r \leq n-1$ in the equation $a = qn + r$ guaranteed by the Division Algorithm.

**Grading:** 1 for giving range of $r$.

/3 5. Suppose suppose $n \geq 2$ is and integer and $\dot{a}, \dot{b} \in \mathbb{Z}_n$. Provide the definition of the sum $\dot{a} + \dot{b}$.

**Solution:** $\dot{a} + \dot{b} = \dot{((a+b) \bmod n)}$

**Grading:** 1 for dot above term on right.

/6 6. Compute $\dot{6} + \dot{6} + \dot{5}$ in $\mathbb{Z}_8$ and justify each step using only the two previous definitions.

**Solution:**

$$\begin{aligned} \dot{6} + \dot{6} + \dot{5} &= \dot{((6+6) \bmod 8)} + \dot{5} \\ &= \dot{4} + \dot{5} \quad \text{since } 12 = (1)8 + 4 \\ &= \dot{(4+5) \bmod 8)} \\ &= \dot{1} \quad \text{since } 9 = (1)8 + 1 \end{aligned}$$

**Grading:** 2 for computing addition separately. 2 for each addition.

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7. Compute the standard form of the complex number $e^{-i\pi/3}$. Your final answer should not contain any trigonometric functions.

   **Solution:** By definition

$$e^{-i\pi/3} = e^{i(-\pi/3)} = \cos(-\pi/3) + i\sin(-\pi/3) = 1/2 - (\sqrt{3}/2)i$$

   **Grading:** 2 for definition of $e^{i(-\pi/3)}$. 2 for trigonometric identity.

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8. Prove by *induction* on $n$ that

$$(\sqrt{3}+i)^n = 2^n(\cos(n\pi/6) + i\sin(n\pi/6))$$

   for all integers $n \geq 1$. Do not use the fact that $(e^{i\theta})^n = e^{in\theta}$. You may use trigonometric identities without proof.

   **Solution:** Base case. Suppose $n = 1$. Then

$$(\sqrt{3}+i)^1 = \sqrt{3}+i = 2(\sqrt{3}/2 + (1/2)i) = 2(\cos((1)\pi/6) + i\sin((1)\pi/6))$$

   and we are done.

   Now assume that $(\sqrt{3}+i)^n = 2^n(\cos(n\pi/6) + i\sin(n\pi/6))$.

   For the induction step we must prove

$$(\sqrt{3}+i)^{n+1} = 2^{n+1}(\cos((n+1)\pi/6) + i\sin((n+1)\pi/6)).$$

$$\begin{aligned}(\sqrt{3}+i)^{n+1} &= (\sqrt{3}+i)^n(\sqrt{3}+i)\\ &= (2^n(\cos(n\pi/6) + i\sin(n\pi/6)))\,(2(\cos(\pi/6) + i\sin(\pi/6)))\end{aligned}$$

   Here we have used the induction assumption and the base case. We continue with

$$\begin{aligned}&= 2^{n+1}(\cos(n\pi/6) + i\sin(n\pi/6))(\cos(\pi/6) + i\sin(\pi/6))\\ &= 2^{n+1}(\cos(n\pi/6)\cos(\pi/6) - \sin(n\pi/6)\sin(\pi/6)\\ &\quad +i(\cos(n\pi/6)\sin(\pi/6) + \sin(n\pi/6)\cos(\pi/6)))\\ &= 2^{n+1}(\cos((n+1)\pi/6) + i\sin((n+1)\pi/6)).\end{aligned}$$

   We used the identities for adding angles in cos and sin in the final step.

   **Grading:** 2 for proof of base case. 1 for induction assumption. 1 for statement of induction step. 2 for correct use of induction assumption in induction step. 4 for remaining proof of induction step.

9. Indicate whether each of the two statements below is true or false. If you think a statement is true, provide some justification. If you think a statement is false, provide a counterexample to it.

   (a) Suppose $n$ is any integer greater than one (not necessarily prime). Then $\mathbb{Z}_n$ satisfies the axiom of commutativity. /3

   **Solution:** True. Suppose $\dot{a}, \dot{b} \in \mathbb{Z}_n$. Then

   $$\dot{a} + \dot{b} = \dot{(a+b) \bmod n} = \dot{(b+a) \bmod n} = \dot{b} + \dot{a}$$

   since the addition of integers is commutative. Similarly,

   $$\dot{a}\dot{b} = \dot{(ab) \bmod n} = \dot{(ba) \bmod n} = \dot{b}\dot{a}$$

   since the multiplication of integers is commutative.

   **Grading:** 1 for "True". 2 for justification.

   (b) There is an integer $n \geq 2$ such that a solution of the equation

   $$x^n - 1 = 0$$

   over the complex numbers has absolute value $n$. /3

   **Solution:** False. Suppose $z \in \mathbb{C}$ is any solution to this equation. Then $z^n = 1$ and taking absolute values of both sides we see that $|z^n| = |1|$. We know that $|z^n| = |z|^n$ and $|z| \geq 0$ so the previous equation tells us that $|z|^n = 1$ for some non-negative real number $|z|$. That implies that $|z| = 1$. We now know that the absolute value of $z$ is 1 and $1 \neq n$ since $n \geq 2$ by hypothesis.

   **Grading:** 1 for "False". 2 for justification.