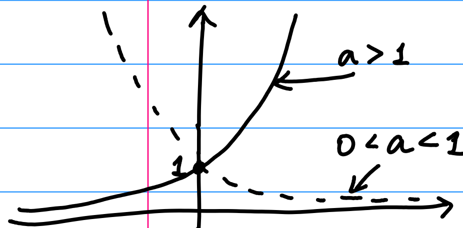
Lecture 1. a^x and LOGs

Sec. 1.1 – 1.4 – HOME
Sec. 1.5 – 1.6

(1) a^x ← x: exponent or power; a: base

Def. a^x, $a>0$

1) $a^0 = 1$
2) $x \in \mathbb{N} \Rightarrow a^x = \underbrace{a \dots a}_{x}$, $\quad a^{-x} = \dfrac{1}{\underbrace{a \dots a}_{x}}$
2) $x \in \mathbb{Q}$, i.e. $x = \dfrac{p}{q} \Rightarrow a^x = \left(\sqrt[q]{a}\right)^p$
3) $x \in \mathbb{R}$, squeeze



Q: where is $a=1$?

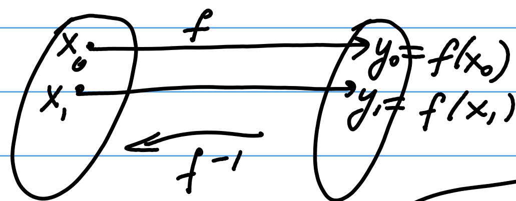
Properties:

1) $a^x \cdot a^y = \underbrace{a \dots a}_{x}\,\underbrace{a \dots a}_{y} = a^{x+y}$
2) $(a^x)^y = (\underbrace{a \dots a}_{x})^y = a^{x \cdot y}$
3) $\dfrac{a^x}{a^y} = a^{x-y}$

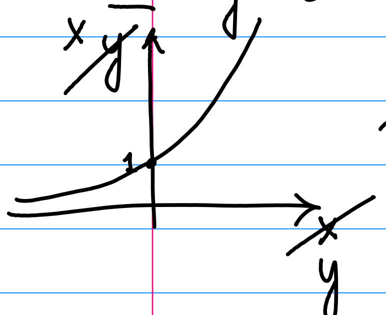
Imp. 1) $a^x > 0$
2) a^x eventually $> x^a$

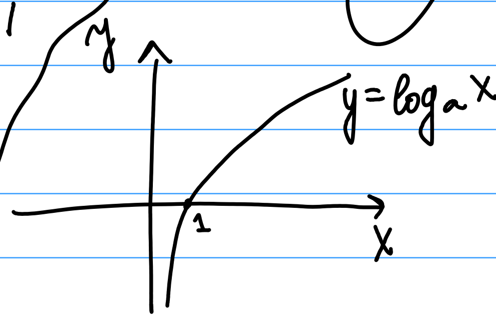
$\boxed{\text{slope} = 1 \Rightarrow y = e^x}$

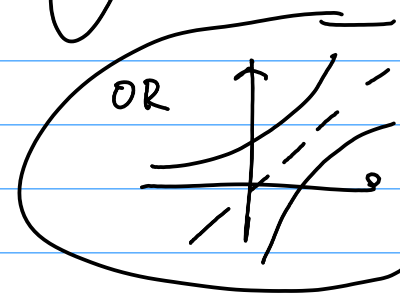
(2) **Inverse fun.** f: 1-1 func.



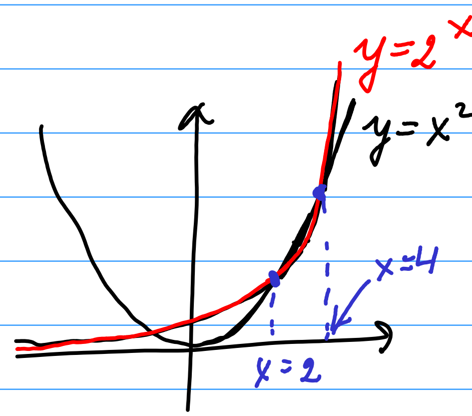
Ex. $y = a^x$ $\quad a>1$







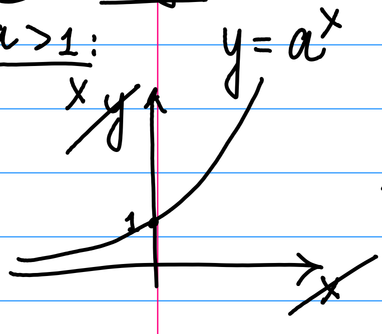
Example: $y = 2^x$
$y = x^2$



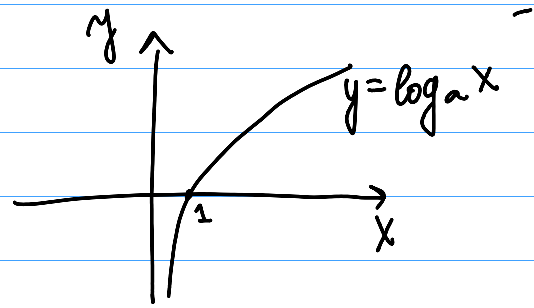
Lecture 2.

③ **Logs** : inverse of $y=a^x \iff x=\log_a y$

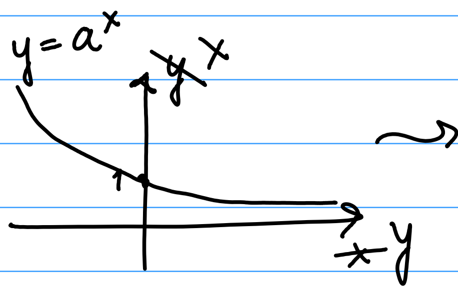
$a>1$:



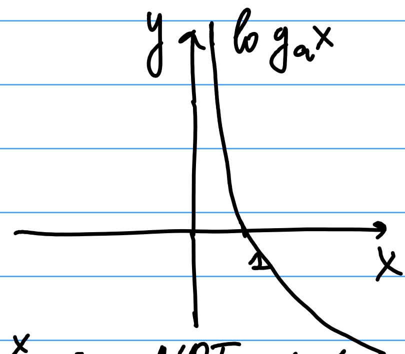
$\rightsquigarrow$



$0<a<1$:



$\rightsquigarrow$



Q: what if $a=1$?

$a=1 \Rightarrow y=a^x$ is NOT 1-1

Properties:

1) $\underbrace{\log_a x}_{t} + \underbrace{\log_a y}_{m} =$

$t: a^t = x \qquad a^m = y \quad =: m$

$x\cdot y = a^{t+m}$

$\Downarrow$

$t+m = \log_a(x\cdot y)$

2) $\log_a x^r = r\log_a x \qquad r\in\mathbb{R}$

WHY? • If $r\in\mathbb{N}$: $\log_a x^r = \log_a(\underbrace{x\ldots x}_{r}) =$ use 1)

• If $r\in\mathbb{Q}$ ...

E.g. $r=\frac{1}{2}$; $\log_a\sqrt{x} = y$

$a^y = \sqrt{x}$

$(a^y)^2 = x$

$a^{2y} = x$

$2y = \log_a x$

$y = \frac{1}{2}\log_a x$ Q.E.D.

Q.E.D.
quot
erat
demonstrandum

3) $\log_a \frac{p}{q} = \log_a(p\cdot q^{-1}) \underset{\text{use 1)}}{=} \log_a p + \log_a q^{-1} = \log_a p - \log_a q$

④ $a = e \Rightarrow \log_e x = \ln x$

Prop. $\boxed{\log_a x = \frac{\ln x}{\ln a}}$

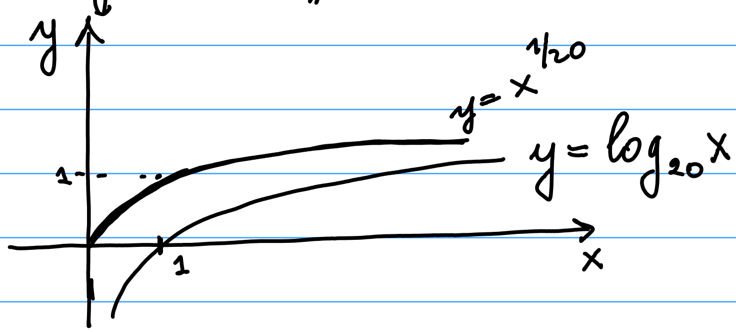
// $\log_a x = y \Leftrightarrow a^y = x$

$y \ln a = \ln x$ //

IMPOT: $\log_a x$ is „smaller" than $\forall\, x^t$, $t>0$

$\forall$ means „any" in math.

Example:

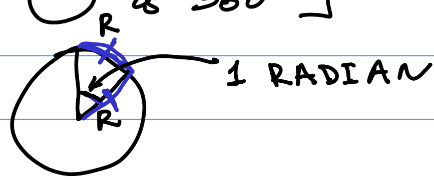


TRIGS. APPEND. D.

HIGH (SCHOOL THINGS)

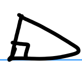
Angles measures:

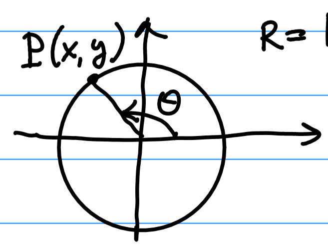
1) degrees [○ is 360°]
2) RADIANS



length of circle $= 2\pi R \Rightarrow$ circle is 2π RADIANS

Trigs defs:

1) For acute angles – via 
2) For obtuse or even negative:



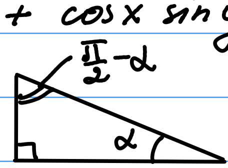
$\begin{cases} y = \sin\theta \\ x = \cos\theta \end{cases}$

Properties: 0) $\sin^2 x + \cos^2 x = 1$

MAIN: 1) $\sin(x+y) = \sin x \cos y + \cos x \sin y$

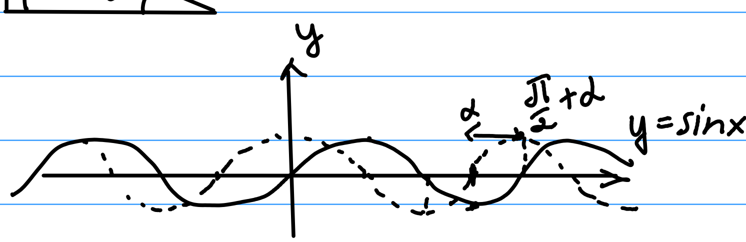
2) $\cos(\frac{\pi}{2} - \alpha) = \sin\alpha$

$\sin(\frac{\pi}{2} - \alpha) = \cos\alpha$



3) $\cos(\frac{\pi}{2} + \alpha) = -\sin\alpha$

$\sin(\frac{\pi}{2} + \alpha) = \cos\alpha$



4) $\cos(x+y) = \cos\left(\underbrace{x+y+\frac{\pi}{2}}-\frac{\pi}{2}\right) \underset{\text{cos is even}}{=} \cos\left(\frac{\pi}{2}-\left(x+y+\frac{\pi}{2}\right)\right) =$

$= \sin(x+\underbrace{y+\frac{\pi}{2}}) = \sin x \cos\left(y+\frac{\pi}{2}\right) + \cos x \cdot \sin\left(y+\frac{\pi}{2}\right) =$

$= \sin x \cdot (-\sin y) + \cos x \cdot \cos y = \cos x \cdot \cos y - \sin x \cdot \sin y$

Lecture 3.

THEN: ✓ $\sin(2x) = 2\sin x \cos x$

✓ $\cos(2x) = \cos^2 x - \sin^2 x = 1 - \sin^2 x - \sin^2 x$

$\cos^2 x - (1-\cos^2 x)$

✓ $\sin(x+y) + \sin(x-y) = 2\sin x \cos y$

$\Downarrow$

„ Product can be re-written as „+" "

✓ $\cos(x-y) + \cos(x-y) = 2\cos x \cos y$

✓ $\cos(x+y) - \cos(x-y) = 2\sin x \sin y$

✓ $\tan^2(x) + 1 = \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x} = \sec^2 x$

✓ $\cot^2(x) + 1 = \csc^2 x$

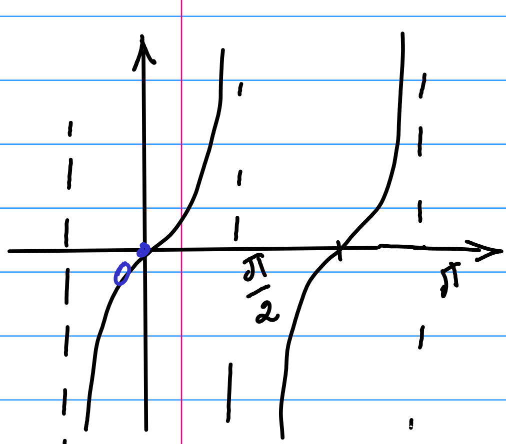
div. by $\cos x, \cos y$

✓ $\tan(x+y) = \dfrac{\sin(x+y)}{\cos(x+y)} = \dfrac{\sin x \cdot \cos y + \sin y \cdot \cos x}{\cos x \cdot \cos y - \sin x \cdot \sin y} =$

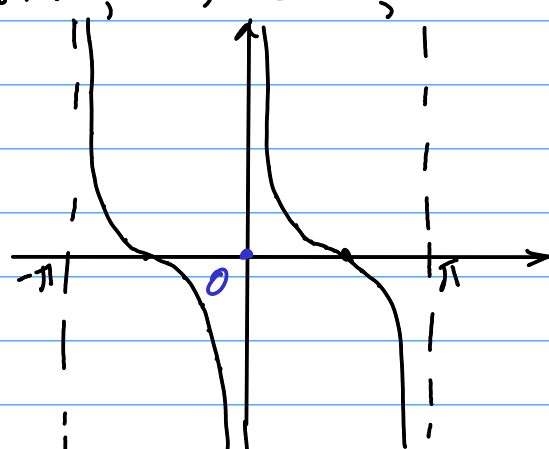
$= \dfrac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$

GRAPHS!

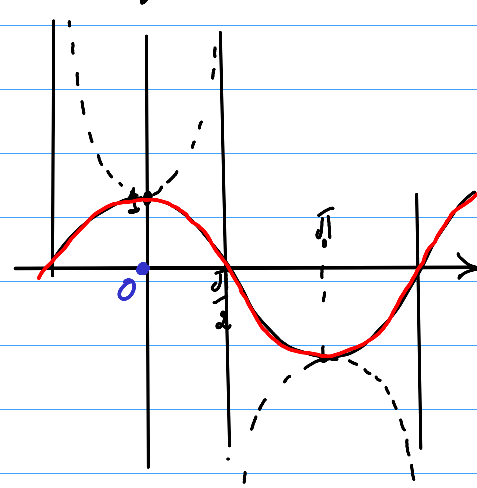
IMPORTANT TO RECALL sin, cos, tan, cot (HOME)



$y = \tan x$



$y = \cot x$



building up $y = \sec x$ from $y = \cos x$

INVERSE TRIGS.

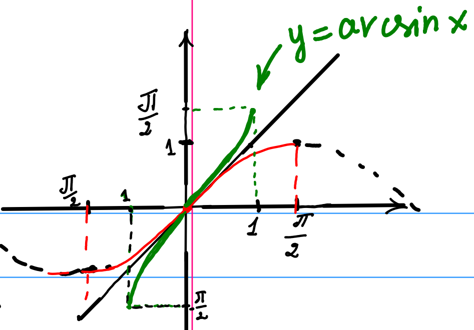
Sec. 1.6.

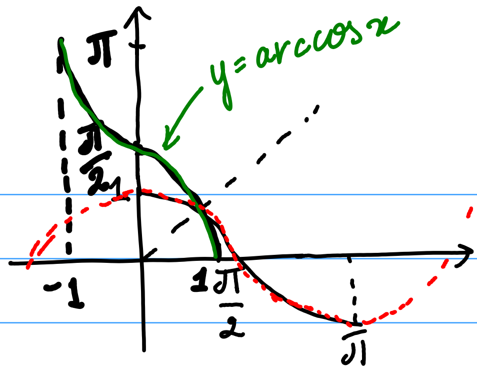
make sin 1−1 → $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

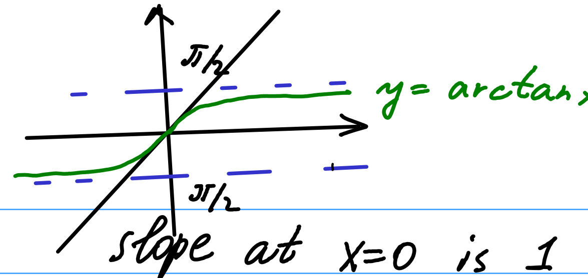
cos 1−1 → $[0, \pi]$

tan 1−1 → $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

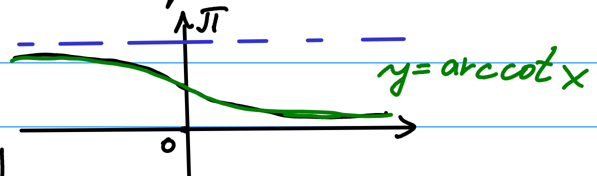
cot 1−1 → $(0, \pi)$







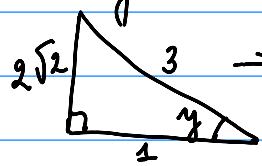
slope at $x=0$ is 1



slope at $x=0$ is 1

TYPICAL PROBLEM (see many in the book) $\sin\left(\arccos\frac{1}{3}\right) = ?$

Let $y = \arccos\frac{1}{3} \Rightarrow \cos y = \frac{1}{3}$



$\rightarrow \sin y = \frac{2\sqrt{2}}{3}$

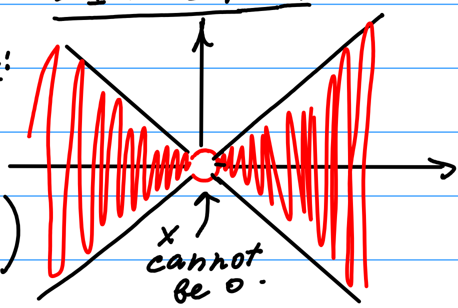
Q.E.D.

HOME: Read Sec. 1.1.–1.6. AND APPENDIX D. → Quiz 1.

Lecture 4. LIMITS.

Sec. 2.2 → at HOME

Motivation:



$y = x\cdot\sin\left(\frac{100}{x}\right)$

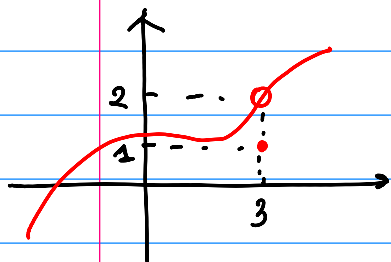
Q: $\lim_{x\to 0} f(x) = ?$

Intuition: 0.

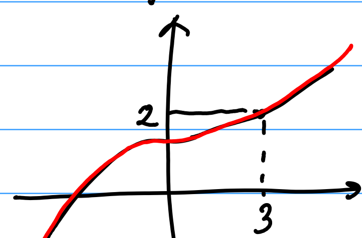
Def. $\lim_{x\to a} f(x) = L$ if one can make $f(x)$ arbitrarily close to L by taking x sufficiently close to a

Note: $f(x)$ can be NOT defined at $x=a$.

Q.

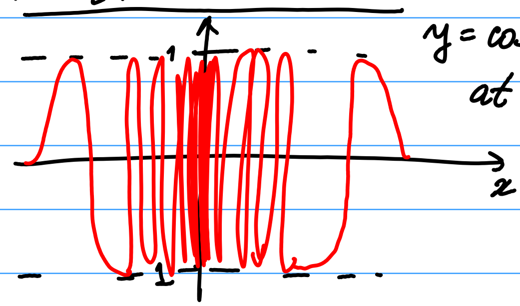


$\lim_{x\to 3} f(x) = ?$



$\lim_{x\to 3} f(x) = ?$

Ex. [NO LIMIT]



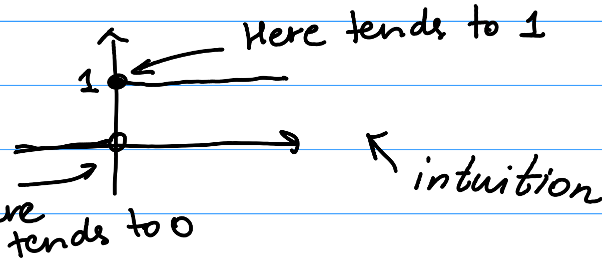
$y = \cos\frac{\pi}{x}$ at $x=0$

One-sided limits:

"Heaviside" function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Q: what is limit at $x=0$?

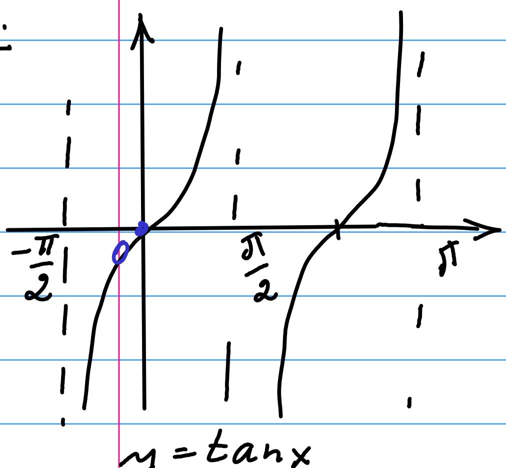


Def: $\lim\limits_{x\to a-} f(x) = L$ if one can make $f(x)$ arbit. close to L by taking x sufficiently close to a from the left

$\lim\limits_{x\to a+} f(x) = L$ - - - - - - → same but

Infinite Limits.

Ex.



Q: [Guess] $\lim\limits_{x\to \frac{\pi}{2}-} \tan x = ?$

$\lim\limits_{x\to \frac{\pi}{2}+} \tan x = ?$

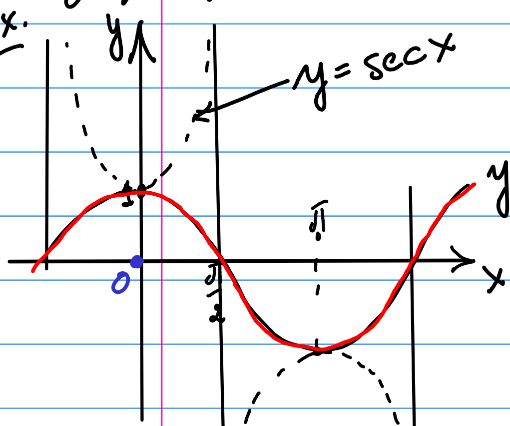
Def. $\lim\limits_{\substack{x\to a\\ a+\\ a-}} f(x) = +\infty$ if we can make $f(x)$ arbitrarily large as $x \to \begin{matrix} a\\ a+\\ a-\end{matrix}$

Def. $\lim\limits_{\substack{x\to a\\ a+\\ a-}} f(x)$ if ... arb. small

Def. [impot.] if $f(x) \xrightarrow[\substack{x\to a\\ a+\\ a-}]{} \pm\infty$ [at least one of these is true]

$\Rightarrow$ the graph of $f(x)$ has vertical asymtote at $x = a$.

Ex.



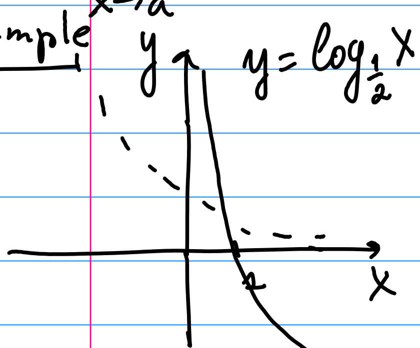
Q: $\lim\limits_{x\to \frac{\pi}{2}-} \sec x = ?$

$\lim\limits_{x\to \frac{\pi}{2}+} \sec x = ?$

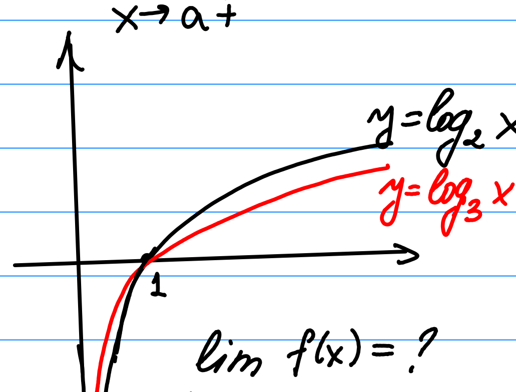
Q. $\lim\limits_{x\to \frac{\pi}{2}} \sec x = ???$

Def. $\lim\limits_{x\to a} f(x) = L \iff \lim\limits_{x\to a+} f(x) = L$ AND $\lim\limits_{x\to a-} f(x) = L$

Example



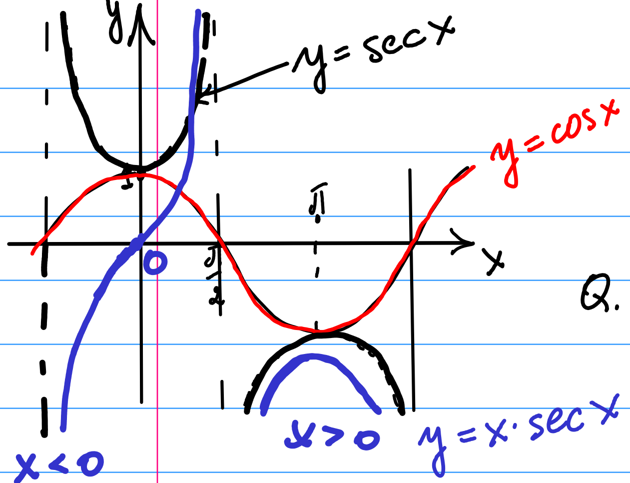
$\lim\limits_{x\to 0+} \log_{\frac{1}{2}} x = ?$



$\lim\limits_{x\to 0+} f(x) = ?$

Lecture 5.

Example [multiplicative intuition]
DIFFICULT



Q: $\lim\limits_{x \to \frac{\pi}{2}+} x \cdot \sec x = ?$

Q.

LAWS OF LIMITS

Theorem: Let $\lim\limits_{x \to a} f(x) = F$, $\lim g(x) = G$ and c is a constant

Then:

① $\lim\limits_{x \to a} (f(x) \pm g(x)) = F \pm G$ ② $\lim\limits_{x \to a} (c \cdot f(x)) = c \cdot F$

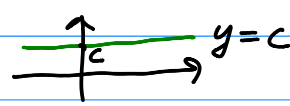
③ $\lim\limits_{x \to a} (f(x) \cdot g(x)) = F \cdot G$ ④ $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G}$ if $G \neq 0$.

Corollary: $\lim\limits_{x \to a} (f(x))^r = (\lim\limits_{x \to a} f(x))^r$ for $r \in \mathbb{R}$

Ex. if $q = \frac{1}{3} \Rightarrow \lim\limits_{x \to a} \sqrt[3]{f(x)} = \sqrt[3]{\lim\limits_{x \to a} f(x)}$

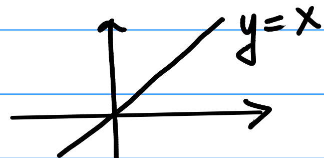
LIMITS OF GOOD FUNCTIONS:

① $y = C$ ← some real number



Q: $\lim\limits_{x \to 0} c = ?$ $\lim\limits_{x \to a} c = ?$

② $y = x$



Q: $\lim\limits_{x \to a} x = ?$

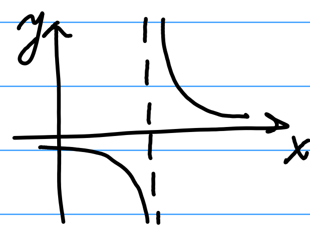
③ $\lim\limits_{x \to a} x^t = ?$ LAWS! $= (\lim\limits_{x \to a} x)^t = a^t$

Example $\lim\limits_{x \to 1} 1001 x^{1002} + 1010 = ?$ 2011

LIMITS OF POLYNOMIALS

Theorem: if $f(x)$ - Polynomial $\Rightarrow \lim\limits_{x \to a} f(x) = f(a)$

Example: $y = \frac{1}{x-2}$



$\lim\limits_{x \to a} \frac{1}{x-2} = ?$ $a \neq 2$

lim at $x = 2 \pm$?

using laws: $\lim\limits_{x\to 1} \frac{1}{x-2} = \frac{1}{\lim\limits_{x\to 1}(x-2)}$ since $\lim\limits_{x\to 1}(x-2) = -1 \neq 0$

⑤ LIMS OF RATIONAL FUNCTIONS

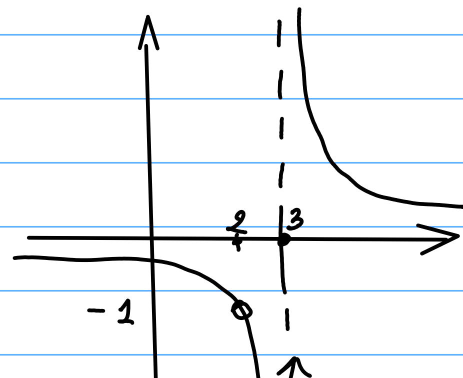
Theorem: if $f(x)$ - rational function, i.e. $f(x) = \frac{\text{Polynomial}}{\text{Polynomial}}$ and a is in the domain then $\lim\limits_{x\to a} f(x) = f(a)$

Example [domains of rational functions]

$f(x) = \frac{1}{(x-2)(x-3)}$ Domain? $x \neq 2$, $x \neq 3$ $\lim\limits_{x\to 5} f(x) = ?$

Example [CANCELING]

$$y = \frac{x-2}{x^2-5x+6}$$



$$y = \frac{1}{x-3}, \quad x \neq 2$$

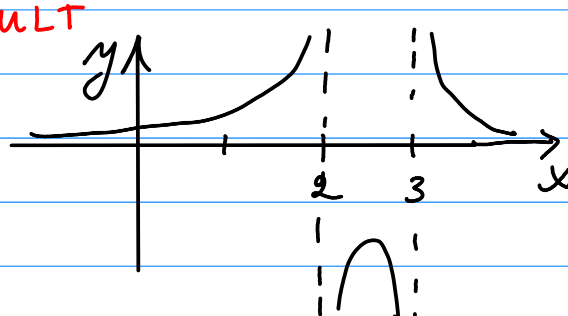
$\lim\limits_{x\to 2} \frac{x-2}{x^2-5x+6} = \frac{1}{2-3} = -1$ → true intuitively see

Example [what if BAD POINT and NO cancelling]

DIFFICULT

$f(x) = \frac{1}{(x-2)(x-3)}$

$\lim\limits_{x\to 3+} f(x) = ?$



Consider $g(x) = (x-2)\cdot(x-3)$

$\lim\limits_{x\to 3+} g(x) = 1 \cdot 0^+$ „a bit larger then 0" = „very small positive number"

NOW $\lim\limits_{x\to 3+} f(x) = \frac{1}{\to \text{very small positive number}}$

$\frac{1}{1}$ $\quad \frac{1}{1/2} = 2$ $\quad \frac{1}{1/3} = 3$ $\quad \frac{1}{1/4} = 4$ $\quad \frac{1}{\text{very small positive}} = \text{very large positive}$

$\Rightarrow \lim\limits_{x\to 3+} f(x) = +\infty$ and we have a vertical asymptote for the graph here.

Example [if BAD point but cancelling possible ...]

$$y = \frac{x-2}{x^2-5x+6} = \frac{1}{x-3}$$

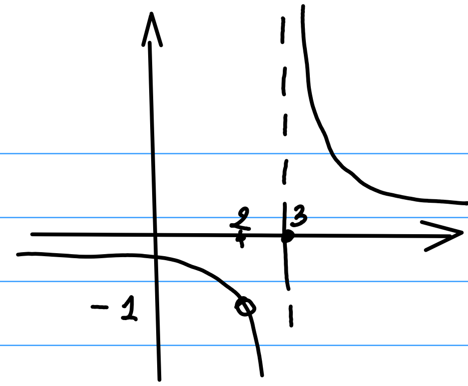
$x \neq 2$

denom. $= 0 \iff \begin{cases} x=2 \\ x=3 \end{cases}$

Q: vertical asymptotes at ??

NOT at $x=2$

only at $x=3$



Lecture 6. Continuation: Computation of lims. [Sec. 2.2–2.3]

Quiz 2: 2.2, 2.3, 2.5. → next Monday **[skip 2.4]**

Last time: cancellation trick:

$$y = \frac{\cancel{x-2}}{(\cancel{x-2})(x-3)} = \frac{1}{x-3}, \quad x \neq 2$$

Example: **[Rationalization trick]** $\lim_{t\to 4} \frac{t-\sqrt{3t+4}}{4-t} = ?$

Recall: $(a+b)\cdot(a-b) = a^2 - b^2$

$$\frac{(t-\sqrt{3t+4})(t+\sqrt{3t+4})}{(4-t)(t+\sqrt{3t+4})} = \frac{t^2-(\sqrt{3t+4})^2}{(4-t)(t+\sqrt{3t+4})} = \frac{t^2-3t-4}{(4-t)(t+\sqrt{3t+4})} =$$

$$= \frac{(\cancel{t-4})(t+1)}{\underbrace{(\cancel{4-t})}_{-1}(t+\sqrt{3t+4})} = -\frac{t+1}{t+\sqrt{3t+4}} \xrightarrow[t\to 4]{} -\frac{4+1}{4+\sqrt{12+4}} = -\frac{5}{8}$$

Example: [**When a „Jump" occurs**]

$$g(y) = \begin{cases} y^2+5, & y < -2 \\ 1-3y, & y \geq -2 \end{cases}$$

1) $\lim_{y\to 0} g(y) = \lim_{y\to 0} (1-3y) = 1-3\cdot 0 = 1$

Because AROUND 0 $g(y) = 1-3y$

Because $1-3y$ is a polynomial

2) $\lim_{y\to -2} g(y) = ?$

On the left „around" point -2:

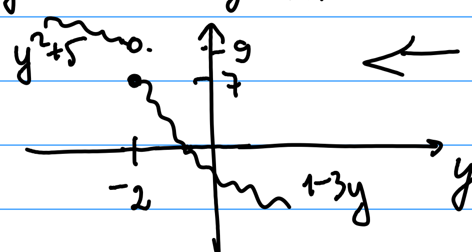
$$\lim_{y\to -2-} g(y) = \lim_{y\to -2-} y^2+5 = (-2)^2+5 = 9$$

Because y^2+5 is a polynomial

On the right:

$$\lim_{y\to -2+} g(y) = \lim_{y\to -2+} 1-3y = 1-3\cdot(-2) = 7$$

$\lim_{y\to -2-} \neq \lim_{y\to -2+} \Rightarrow \lim_{y\to -2}$ does not exist!



← very approximate sketch

only shows behavior at $x=-2$

Lecture 7.

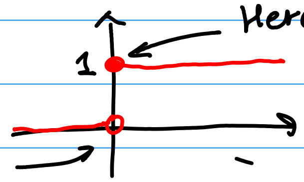
Comparison Theorem If $f(x) \le g(x)$ around a ($x=a$ – whatever) $\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$.

Squeeze Theorem [very IMPORT.] If $f(x) \le g(x) \le h(x)$ for x around a, but $x=a$ - whatever.

AND $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x)$ then $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x)$

Example: Recall Heaviside function: $H(x) = \begin{cases} 0, & x<0 \\ 1, & x \geq 0 \end{cases}$

We know:



i.e. $\lim\limits_{x\to 0+} H(x) = 1$ $\lim\limits_{x\to 0-} H(x) = 0$.

$\Rightarrow \lim\limits_{x\to 0} H(x)$ does not exist

Q. $\lim\limits_{x\to 0} x^2 \cdot H(x) = ?$ We cannot do it as $\lim\limits_{x\to 0} x \cdot \lim\limits_{x\to 0} H(x)$ as

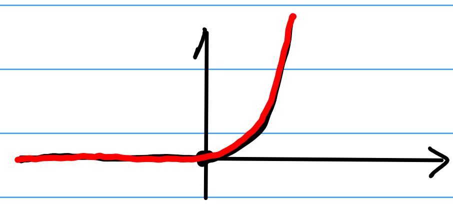
$0 \le H(x) \le 1$ for all x near 0.

$0 \le x^2 \cdot H(x) \le x^2$ $\Rightarrow \lim\limits_{x\to 0} x^2 \cdot H(x) = 0$

$0 \leftarrow$ same $\rightarrow 0$ ($x \to 0$)

this corresponds to our INTUITION:

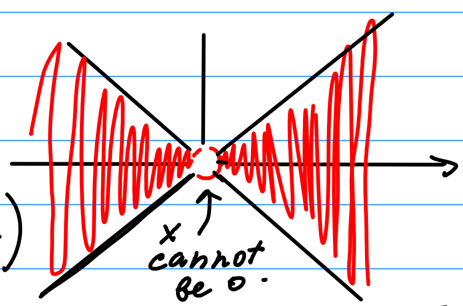
intuition: $x^2 \cdot H(x) = \begin{cases} 0, & x<0 \\ x^2, & x \geq 0 \end{cases}$



Example: difficult:

Recall this function

$y = x \cdot \sin\left(\frac{100}{x}\right)$



Intuition: $\lim\limits_{x\to 0} x \cdot \sin\frac{100}{x} = 0$

How to compute?

Attempt 1: $\lim\limits_{x\to 0}\left(x \cdot \sin\frac{100}{x}\right) =$

$= \boxed{\lim\limits_{x\to 0} x} \cdot \boxed{\lim\limits_{x\to 0} \sin\frac{100}{x}} = ?$

$\underbrace{\lim\limits_{x\to 0} x}_{=0}$; $\lim\limits_{x\to 0} \sin\frac{100}{x}$ → no limit.

Now try to use „Squeeze theorem":

$-1 \le \sin\frac{100}{x} \le 1$

if $x > 0 \Rightarrow -x \le \sin\frac{100}{x} \le x \Rightarrow \lim\limits_{x\to 0+} x \cdot \sin\frac{100}{x} = 0$

if $x < 0 \Rightarrow -x \geq \sin\frac{100}{x} \geq x \Rightarrow \lim\limits_{x\to 0-} x \cdot \sin\frac{100}{x} = 0$

Since $\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0-} f(x) \Rightarrow \lim\limits_{x\to 0} f(x) = 0$. Q.E.D.

Continuity. (We finally define „good functions".)

<u>Def</u> $f(x)$ is continious at $x=a$ if $\lim_{x\to a} f(x) = f(a)$

Otherwise: $f(x)$ is DISCONT. at $x=a$

<u>Def.</u> $f(x)$ is cont. on $[a,b]$ if it is cont. at every $x=c$, $c\in[a,b]$

* if $f(x)$ is not defined on the left of a, then we require only $\lim_{x\to a^+} f(x) = f(a)$ and say „cont. from the right".

<u>Roughly:</u> $f(x)$ is cont. on $[a,b]$ if we can draw the graph from start to finish without ever once picking up our pencil.

$\Rightarrow$ Q: which functions are continious at each point of their domain? [Recall the graphs]

$\sin x$, $\cos x$, $\arcsin x$, $\arccos x$, $\arctan x$, a^x, e^x, $\sqrt{x}$

Q: polynomials?

<u>Example:</u> $f(x) = x^{2011} + 2010$

Q: is it cont. at $x=1$?

<u>Solution:</u> 1) We know that for polynom. lim exists and $\lim_{x\to 1}(x^{2011}+2010) = 2011$

2) $f(1) = 2011 \Rightarrow \lim_{x\to 1}(x^{2011}+2010) = f(1)$

$\Rightarrow$ cont. at $x=1$. $\rightarrow$ the same for every poly $\rightarrow$

<u>Theorem:</u> Polynomials are cont. at every point.

We could proceed the same way for other functions but instead:

<u>**Laws of Continuaty:**</u> let $f(x)$ and $g(x)$ are cont. at $x=a$ and c is a constant, then.

$f(x)+g(x)$, $f(x)-g(x)$, $f(x)\cdot g(x)$, $\dfrac{f(x)}{g(x)}$, $g(a)\neq 0$, $c\cdot f(x)$ are cont. at $x=a$.

<u>Proof.</u> very easy $\rightarrow$ see p.121 //

<u>Corollary:</u> 1) Rational functions are cont. at every point of their domain.

2) $\tan x = \dfrac{\sin x}{\cos x}$, $\cot x$, $\sec x$, $\csc x$ are cont. at every point of their domain

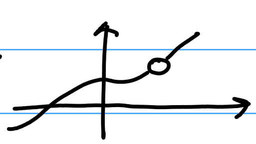
<u>Lecture 8.</u> talking about $\tan x$

let's talk about „bad" points.

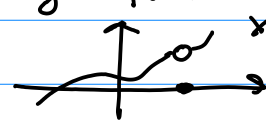
THREE TYPES of DISCONT:

1) Removable: when it can be removed by defining $f(a) = \lim_{x\to a} f(x)$.

Ex.

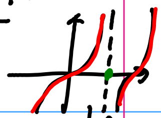


OR



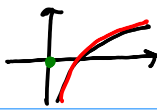
2) Infinite discont.

Q: invent here more ex.



$y = \tan x$

OR

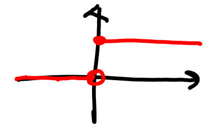


$y = \log_a x$

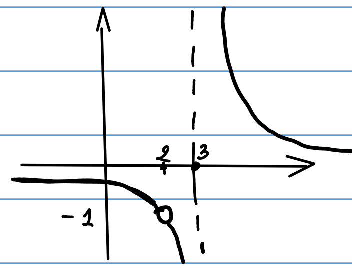
also a^x

3) „jump" discont.

Ex. Heaviside $H(x)$



Example: $y = \dfrac{x-2}{(x-2)(x-3)} = \dfrac{1}{x-3}, \; x \neq 2$



Q: $x=2$? $x=3$?

→ infinite discont. at $x=3$

→ removable at $x=2$

Theorem: [IMPORT] If $\lim\limits_{x\to a} g(x) = b$ and $f(x)$ is cont. at $x = b$ then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$$

Example: $\lim\limits_{x\to 0} e^{\sin x} = e^{\lim\limits_{x\to 0} \sin x} = e^0 = 1$

because whatever $\lim\limits_{x\to 0} \sin x$ is going to be, $y = e^x$ is continious for all $x \in \mathbb{R}$.

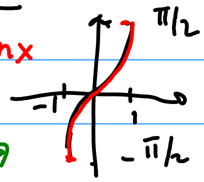
Theorem: If g is cont. at a, and f is cont. at $g(a)$, then $f(g(x))$ is cont. at a.

Example: Q: prove that $e^{\sin x}$ is cont at $x=0$.

$f = e^x, g = \sin x, \; a = 0, \; f = e^x$ cont at $g(0) = 0$ Q.E.D.

Example: where is $\arcsin(x+1)$ continious?

$y = \arcsin x$



1) $f = \arcsin(x)$ is cont. everywhere in its domain
$g = x + 1$ – also

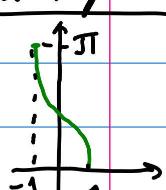
$\Rightarrow$ $f(g(x))$ cont. everywhere in its domain.

2) domain: $x + 1 \to$ all $\mathbb{R}$

$\arcsin x \to$ only $[-1, 1]$

So, $f(g(x))$ is defined when $-1 \le x+1 \le 1$, i.e. when $-2 \le x \le 0$

Example: where is $\arccos(e^x + 1)$ continous?



$y = \arccos x$

1) $f = \arccos(x)$ is cont. everywhere in its domain
$g = e^x + 1$ – also

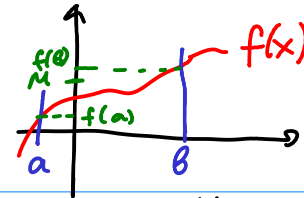
$\Rightarrow$ $f(\;(x))$ cont. ever where in its domain.

2) domain: $e^x + 1 \to$ all $\mathbb{R}$

$\arccos x \to [-1, 1]$

So, $f(g(x))$ is defined when $-1 \le e^x + 1 \le 1$, i.e. $-2 \le e^x \le 0 \to$ NEVER!

Intermediate value thm.

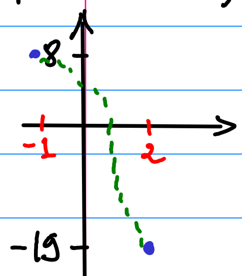


THEOREM: let $f(x)$ be continious on $[a,b]$, $f(a) \neq f(b)$, and M - between $f(a)$ and $f(b)$ then there exists such value of x, $x=c$ that $a \le c \le b$ and $f(c) = M$.

ROUGHLY speaking, the thm. says that „a cont. function on $[a,b]$ takes all values between $f(a)$ and $f(b)$"

How to use it? Ex. (easy) show that $\underbrace{p(x) = 2x^3 - 5x^2 - 10x + 5}_{\text{cont.}}$ has roots on $[-1,2]$

1) $p(-1) = 8$, $p(2) = -19$ $\Rightarrow$ $p(-1) \neq p(2)$



2) $-19 \le 0 \le 8$ $\Rightarrow$ there exists $c \in [-1,2]$ s.t. $p(c) = 0$ $\rightarrow$ root.

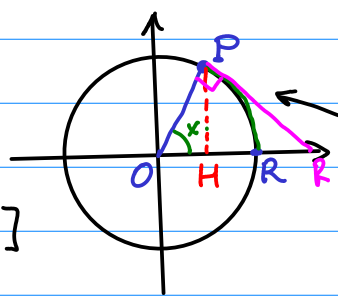
Lecture 9. **Example:** $f(x) = 20 \cdot \sin(x) \cdot \cos(8x)$ does $f(x) = 10$ for some x, $0 \le x \le \frac{\pi}{2}$?

1) $\sin(x)$ and $\cos(8x)$ – cont. funct. for all real x.
$\Rightarrow$ their PRODUCT – also

2) $f(0) = 20 \cdot \sin(0) \cdot \ldots = 0$
$f(\pi/2) = 20 \cdot 1 \cdot \cos(4\pi) = 20 \cdot \cos(0) = 20$.
$\Rightarrow$ f takes every value between 0 and 20 $\Rightarrow$ including 10.

Theorem: $$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

1) trig. circle:



$x \in [0, \frac{\pi}{2}]$

x in Rad $\rightarrow$ length of the arc.

$|PH| = \sin x$

see that $|PH| < |PR| = \theta < |PR|$

$\underline{\sin x < x < \tan x}$

2) divide by $\sin x$! $1 < \frac{x}{\sin x} < \sec x$

$x \to 0+$: $1 \to 1$; $x \to 0+$ ← only because we assumed $x \in [0, \frac{\pi}{2}]$: $\sec x \to 1$

$\Rightarrow$ by Squeeze thm $\lim_{x\to 0+} \frac{x}{\sin x} = 1$.

3) $\lim_{x\to 0-} \frac{\sin x}{x} = ?$

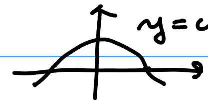
let $t=-x$, then when $x\to 0-$, $t\to 0+$ 4) $\Rightarrow$

$$\lim_{x\to 0-}\frac{\sin x}{x}=\lim_{t\to 0+}\frac{\sin(-t)}{-t}=\lim_{t\to 0+}\frac{\sin t}{t}=1$$

$\lim_{x\to 0}\frac{\sin x}{x}=1$

Q.E.D.

Theorem. $\lim_{x\to 0}\frac{\cos x-1}{x}=0$



Proof. 1) Assume that $-\frac{\pi}{2}<x<\frac{\pi}{2}$

$\Rightarrow \cos x>0 \Rightarrow \cos x=\sqrt{1-\sin^2 x}$

2) then $\lim_{x\to 0}\frac{\cos x-1}{x}=\lim_{x\to 0}\frac{\sqrt{1-\sin^2 x}-1}{x}=\lim_{x\to 0}\frac{(\sqrt{1-\sin^2 x}-1)\cdot(\sqrt{1-\sin^2 x}+1)}{x\cdot(\sqrt{1-\sin^2 x}+1)}=$

$$=\lim_{x\to 0}\frac{-\sin^2 x}{x\cdot(\sqrt{1-\sin^2 x}+1)}=\underbrace{\lim_{x\to 0}\frac{\sin x}{x}}_{=1}\cdot\lim_{x\to 0}\frac{\overset{\to 0}{-\sin x}}{\underbrace{(\sqrt{1-\sin^2 x}+1)}_{\to 1}}=0.$$

Example. $\lim_{x\to 0}\frac{\tan 4x}{x}$

$$\lim_{x\to 0}\frac{\tan 4x}{x}=\lim_{x\to 0}\left(\underbrace{\frac{\sin 4x}{4x}}_{\to 1}\cdot\frac{4}{\underbrace{\cos x}_{\to 1}}\right)=4$$

lecture 10. Limits at ∞.

Def. $\lim_{x\to+\infty}f(x)=L$ $(x\to\infty)$ if
1) f is defined for all $x>a$, a – some number
2) $f(x)$ is arb. close to L if x is suff-ly large.

Def. $\lim_{x\to-\infty}f(x)=L$ if
1) f is defined for all $x<a$, a – some number
2) $f(x)$ is arb. close to L if x is suff-ly small.

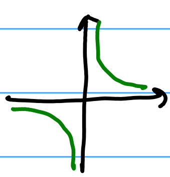
Def. if either of the above is true then $y=L$ is a **horizontal asymptote.**

Ex. $\arctan x$

$\lim_{x\to+\infty}=\frac{\pi}{2}$ $\Rightarrow$ horizontal asymptotes: $y=\pm\frac{\pi}{2}$

$\lim_{x\to-\infty}=-\frac{\pi}{2}$

Ex. $\frac{1}{x}$



$\lim_{x\to+\infty}\frac{1}{x}=0$ $\qquad \lim_{x\to-\infty}\frac{1}{x}=0$

Ex. $\frac{1}{x^{99}}$ $\qquad \lim_{x\to+\infty}\frac{1}{x^{99}}=\frac{1}{\text{Large}}=0.$

$\lim_{x\to-\infty}=\frac{1}{\text{Large, negative}}=0$! $\frac{1}{-1}=-1$ $\quad \frac{1}{-2}=-\frac{1}{2}$ $\quad \frac{1}{-3}=-\frac{1}{3}$

Ex. $\frac{1}{x^{1/2}}$ $\qquad \lim_{x\to+\infty}=\frac{1}{\text{Large}}=0$ $\qquad \lim_{x\to-\infty}\frac{1}{\sqrt{\text{negative}}}$ = does not exist

Lecture 11. Sec. 2.6.

Recall that $\lim_{x\to\infty}\frac{1}{x^{99}} = \lim_{x\to-\infty}\frac{1}{x^{99}} = 0$, but $\lim_{x\to-\infty}\frac{1}{x^{1/2}}$ DNE

Th If $r>0$, rational number, then $\lim_{x\to\infty}\frac{1}{x^r}=0$
if $r>0$, rational number, s.t. x^r is defined for $x<0$, then
$$\lim_{x\to-\infty}\frac{1}{x^r}=0$$

Polynomials at $\pm\infty$

Ex. $\lim_{x\to\infty}(2x^4-x^2)$

Will „plug-in" work? $\lim_{x\to\infty}(2x^4-x^2) = \underline{\infty-\infty} = 0\ ?$ NO!
← good sign that you should try another method

$$\lim_{x\to\infty}(2x^4-x^2-8x) = \lim_{x\to\infty}\left(x^4\cdot\left(2-\frac{1}{x^2}\right)\right) = \infty\cdot 2 = \infty$$

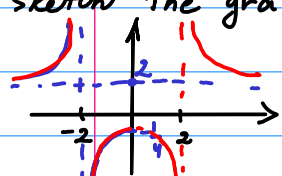
Ex. $\lim_{x\to-\infty}(-\frac{1}{3}t^5+t)$ Again, if simply plug, then $\lim = \infty-\infty = ?\ 0?$ no!

$$\lim_{x\to-\infty}\left(\frac{1}{3}t^5-t\right) = \lim_{t\to-\infty}\left(t^5\left(-\frac{1}{3}-\frac{1}{t^4}\right)\right) = -\infty\cdot\left(-\frac{1}{3}\right) = \infty$$

Rational Functions at $\pm\infty$

Ex. [deg(num) = deg(den.)]
$$\lim_{x\to\infty}\frac{2x^2+1}{x^2-4} = \lim_{x\to\infty}\frac{\cancel{x^2}(2+\frac{1}{x^2})}{\cancel{x^2}(1-\frac{4}{x^2})} = 2$$
$$\lim_{x\to-\infty}\frac{2x^2+1}{x^2-4} = \lim_{x\to\infty}\frac{\cancel{x^2}(2+\frac{1}{x^2})}{\cancel{x^2}(1-\frac{4}{x^2})} = 2$$

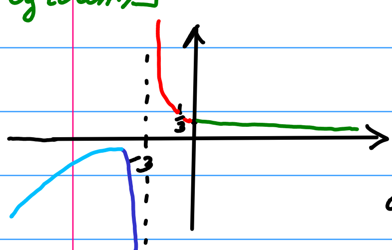
to sketch the graph figure out behaviour at $x=\pm2$: $\lim_{x\to2+} = \frac{2\cdot2^2+1}{0+} = +\infty$, $\lim_{x\to2-} = \frac{2\cdot2^2+1}{0-} = -\infty$



← horizontal asymptote $y=2$

function even: $f(-x)=f(x) \Rightarrow$ graph is symmetrical w.r.t. Oy.

Ex. [deg(num.) > deg(den.)]
$$\lim_{x\to\infty}\frac{2x^4+1}{x+3} = \lim_{x\to\infty}\frac{x^4\left(2+\frac{1}{x^4}\right)}{x\left(1+\frac{3}{x}\right)} = \lim_{x\to\infty}x^3\cdot\frac{2}{1} = \infty\cdot2 = \underline{\infty}$$



How to sketch the graph of such function?
- $x=0 \to y=\frac{1}{3}$
- $x=-3$ is not from the domain → check it using one-side limits:

1) $\lim_{x\to-3+}\frac{2x^4+1}{x+3} = \frac{2\cdot3^4+1}{0+} = \infty$

2) $\lim_{x\to-3-}\frac{2x^4+1}{x+3} = \frac{2\cdot3^4+1}{0-} = -\infty$

Now to finish the sketch, we need to figure out what happens as $x\to-\infty$
$$\lim_{x\to-\infty}\frac{2x^4+1}{x+3} = \lim_{x\to-\infty}(x^3\cdot2) = \underline{-\infty}$$

Ex. [deg(num) < deg(den)]
$$\lim_{x\to\infty}\frac{x+3}{2x^4+1} = \lim_{x\to\infty}\frac{x(1+3/x)}{x^4(1+1/x^4)} = \lim_{x\to\infty}\frac{1}{x^3}\cdot1 = 0.$$

Other fun. at ∞

Ex. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{3x^2+6}}{5-2x} = \lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2\left(3+\frac{6}{x^2}\right)}}{x\left(\frac{5}{x}-2\right)} \overset{Q.}{=} \lim\limits_{x\to-\infty} \dfrac{|x|\cdot\sqrt{3+\frac{6}{x^2}}}{x\,(5/x-2)} \circleddash$

Since $x\to-\infty \Rightarrow x<0$ over there $\Rightarrow |x|=-x \Rightarrow \circleddash -\dfrac{\sqrt3}{-2} = \dfrac{\sqrt3}{2}$

Lecture 12.

Ex. $\lim\limits_{x\to\infty}\left(\sqrt{x^4+x^2}-x^2\right) = ?$

Idea 1: $\lim\limits_{x\to\infty}\left(\sqrt{x^4+x^2}-x^2\right) = \infty-\infty \leftarrow$ BAD

Idea 2: $\lim\limits_{x\to\infty}\left(\sqrt{x^4(1+1/x^2)}-x^2\right) \underset{\substack{x^2>0\\ \text{always}}}{=} \lim\limits_{x\to\infty}\left(x^2\cdot\sqrt{1+1/x^2}-x^2\right) = x^2\cdot\lim\limits_{x\to\infty}\underbrace{\left(\overbrace{\sqrt{1+1/x^2}}^{\to1}-1\right)}_{\to 0} = \infty\cdot0$ BAD

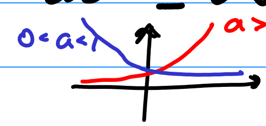
Idea 3: $\lim\limits_{x\to\infty}\left(\sqrt{x^4+x^2}-x^2\right) = \lim\limits_{x\to\infty}\dfrac{\left(\sqrt{x^4+x^2}-x^2\right)\cdot\left(\sqrt{x^4+x^2}+x^2\right)}{\left(\sqrt{x^4+x^2}+x^2\right)} =$

$= \lim\limits_{x\to\infty}\dfrac{x^4+x^2-x^4}{x^2\left(\sqrt{1+1/x^2}+1\right)} = \lim\limits_{x\to\infty}\dfrac{1}{\left(\underbrace{\sqrt{1+1/x^2}}_{\to1}+1\right)} = \dfrac12$ good!

a^x and logs and trigs at $\pm\infty$

Ex. $a>1$: $\lim\limits_{x\to\infty} a^x = \infty$, $\lim\limits_{x\to-\infty} a^x = 0.$

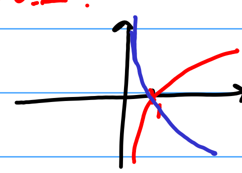
Ex. $0<a<1$: $\lim\limits_{x\to\infty} a^x = 0$, $\lim\limits_{x\to-\infty} a^x = \infty$



Remark: Q: formally speaking, does the limit exists?
A: No, the limit does not exist, ∞ - is a special type of discontinuity. In the exams; write ∞ $(-\infty)$, not „D.N.E."

Ex. $a>1$: $\lim\limits_{x\to\infty}\log_a x = \infty$, $\lim\limits_{x\to-\infty}\log_a x = $ D.N.E.

Ex. $0<a<1$: $\lim\limits_{x\to\infty}\log_a x = -\infty$, $\lim\limits_{x\to-\infty}\log_a x = $ D.N.E.



Ex. $\lim\limits_{x\to\infty}\sin x \overset{Q}{=}$ DNE $\qquad \lim\limits_{x\to\infty}\cos x =$ DNE

$\lim\limits_{x\to-\infty}\sin x \overset{Q}{=}$ DNE $\qquad \lim\limits_{x\to-\infty}\cos x =$ DNE

Sec. 2.6. finished.
skip precise def. at the end of Sec. 2.6.

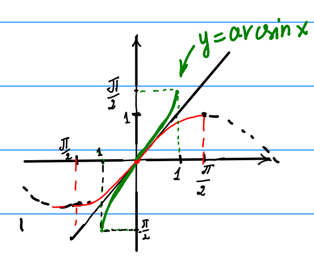
Sec. 2.7 and 2.8. Derivatives.

Def. Tangent line to a graph at point P is a line that „just touches" the graph at P.

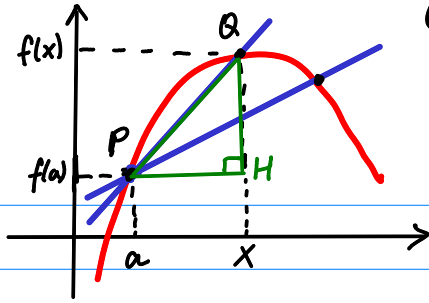
WHY use tangents: "I told you that tangent line to $y=\sin x$ is $y=x$ at $P=(0,0)$, i.e. the slope is 1.

Note: if line is $y=mx+b$, then m - slope.

So, it was helpful.



Consider



Compute the slope of the tangent line at point P.

Consider all lines PQ.

the slope of any of them is:

$$m = \tan\angle QPH = \frac{|QH|}{|PH|} = \frac{f(x)-f(a)}{x-a}$$

Line PQ is tangent if P is very close to Q, i.e. $x \to a$

m - slope of the tangent at point P: $\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}$

Equivalently: if $h = x-a$, then if $x \to a$, then $h \to 0$ and

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(h+a)-f(a)}{h}$$

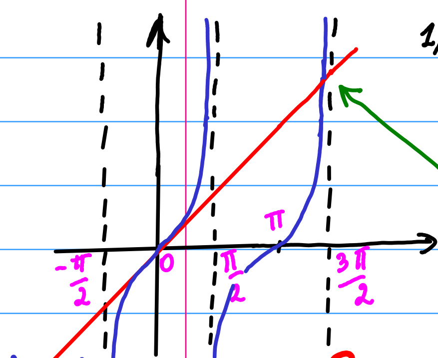
Def. If lim ↗ exists then this limit is **the derivative of f(x) at point a: $f'(a)$.**

Ex. $m_0 = \sin'(0) = \lim\limits_{x\to 0} \frac{\sin(x)-\sin(0)}{x-0} = \lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

1) tangent line is of the form: $y = m_0 \cdot x + b \Rightarrow y = x + b$
2) $(0,0)$ should belong to the line $y = x + b \Rightarrow 0 = 0 + b \Rightarrow b = 0$

$\Rightarrow y = \sin x$ has at $(0,0)$ tangent line $y = x$

Ex. $\tan'(0) = \lim\limits_{x\to 0} \frac{\tan(x)-\tan(0)}{x-0} = \lim\limits_{x\to 0} \frac{\sin x}{x} \cdot \lim\limits_{x\to 0} \frac{1}{\cos x} = 1 \cdot 1 = 1$



1) tangent line is of the form: $y = m_0 \cdot x + b \Rightarrow y = x$
2) $(0,0)$ should belong to the line $y = x + b \Rightarrow b = 0$

$\Rightarrow y = \tan x$ has tangent line $y = x$ at $(0,0)$.

Remark! tangent line **can cross** the graph in **several points** but only one common point in „around" point P

Lecture 12. **Sec. 2.7.-2.8.**

Quiz 3 will be on Sec. 2.6, 2.7, 2.8

General Formula for tangent line at $x=a$! Read 3.1, 3.2 !

$y = f'(a) \cdot x + b$
↑ slope

When $x = a$ graphs $y = f(x)$ and of tangent must be intersected $\Rightarrow$

$$f(a) = f'(a) \cdot a + b \Rightarrow b = f(a) - f'(a) \cdot a$$

$$y = f'(a) \cdot x + f(a) - f'(a) \cdot a \rightarrow y - f(a) = f'(a) \cdot (x-a)$$

Derivative as a function plug x instead of a:

Def. Derivative of function $f(x)$ is $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$

Def. $f(x)$ is **differentiable at a** if $f'(a)$ exists.
$f(x)$ is **differentiable on (a,b)** $\left[\begin{matrix} a \text{ can be } -\infty \\ b \text{ can be } +\infty \end{matrix}\right)$ if $f'(x)$ is diff. at every its point.

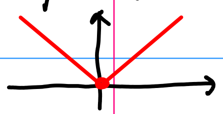
Thm. if $f(x)$ is differentiable at a then it is CONT. at a.

$\Rightarrow$ if function is NOT cont. at a $\Rightarrow$ it is not differentiable at a.

Remark: CAN be CONT. but NOT DIFF.

Ex. $f(x)=|x|$ at $x=0$: $f'(0)$: 1) $\lim\limits_{h\to 0+} \dfrac{|0+h|-|0|}{h} = \lim\limits_{h\to 0+}\dfrac{h}{h} = 1$

2) $\lim\limits_{h\to 0-} \dfrac{|0+h|-|0|}{h} = \lim\limits_{h\to 0-} \dfrac{-h}{h} = -1$ $\Rightarrow$ $f'(0)$ DNE



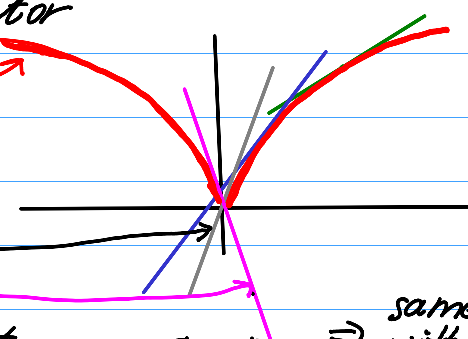
Notation the followings, $y=f(x)$ mean the same: $f'(x) = y' = \dfrac{df}{dx} = \dfrac{dy}{dx} = \dfrac{d}{dx}(f(x)) =$

$= \dfrac{d}{dx}(y) = Df(x) = D_x f(x)$; $f'(a) = \dfrac{df}{dx}\Big|_{x=a}$

differentiation operator

When derivative exists?

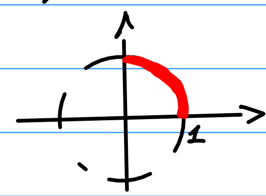
I. Consider function



From the right tangent lines tends to grey line
From the left — " — " — " — " — " — pink — " —
$\Rightarrow$ different slopes (= limits) $\Rightarrow$ limit (slope) at $x=0$ DNE. $\Rightarrow$ same with $y=|x|$

„corner" or „kink"

II. Ex. $x^2+y^2=1$ for $\begin{matrix} x\ge 0 \\ y\ge 0\end{matrix}$ $\Rightarrow$



$y=f(x)$, $f(x)=\sqrt{1-x^2}$

$$\frac{df}{dx}\Big|_{x=1} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\sqrt{1-(1+h)^2}-\overbrace{\sqrt{1-1^2}}^{0}}{h} =$$

$$= \lim_{h\to 0}\frac{\sqrt{1-1-2h-h^2}}{h} = \lim_{h\to 0}\frac{\sqrt{h(-2-h)}}{h}$$

If $h>0 \Rightarrow h=\sqrt{h^2}$ and $\lim\limits_{h\to 0+}\dfrac{\sqrt{h(-2-h)}}{\sqrt{h^2}} =$

$= \lim\limits_{h\to 0+}\sqrt{\dfrac{-2}{h}-1} = \text{DNE} \Rightarrow \dfrac{df}{dx}\Big|_{x=1}$ DNE

If $h<0 \Rightarrow h=-\sqrt{h^2}$ and $\lim\limits_{h\to 0-}\dfrac{\sqrt{h(-2-h)}}{-\sqrt{h^2}} = \lim\limits_{h\to 0-} -\sqrt{\underbrace{\dfrac{-2}{h}}_{\to+\infty}-1} =$ $= -\infty$

$\left(\dfrac{-2}{h} \to -\infty \text{ for } h\to 0+\right)$

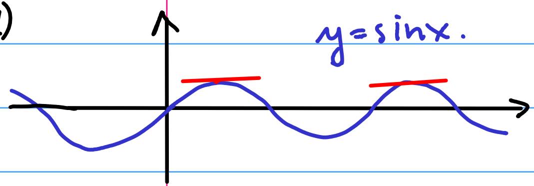
„when vertical asymptote"

III When $f(x)$ is discont.

High Derivatives.

$f'(x)$ – is a function $\Rightarrow$ has its own derivative $f''(x)$, in its turn —"— $f'''(x)$, ..., $f^{(n)}(x)$ – n^{th} derivative.

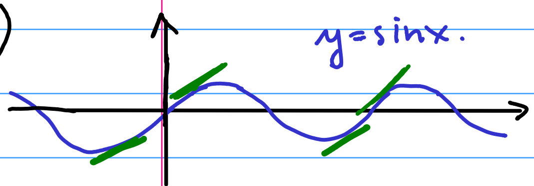
Sketching Derivatives.

1) 

what is the slope of tangent lines here?

angle $= 0 \Rightarrow \tan 0 = 0 \Rightarrow$ slope $= 0$

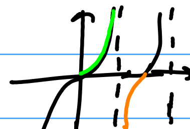
$\Rightarrow f'(\text{these points}) = 0.$

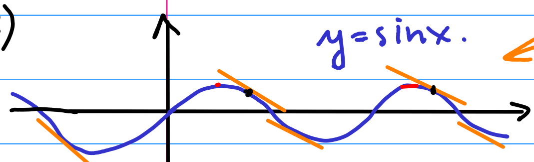
2) 

slopes? $0 < \text{angle} < \frac{\pi}{2}$

$\Rightarrow \tan > 0$

$\Rightarrow$ slope $m > 0 \Rightarrow f'(\text{at these points}) > 0$

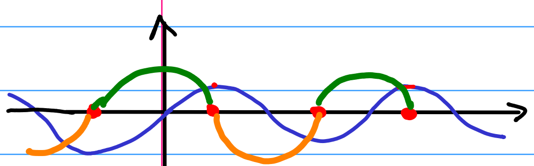


3) 

slopes? $\frac{\pi}{2} < \text{angle} < \pi$

$\Rightarrow \tan < 0 \Rightarrow m < 0 \Rightarrow f' < 0$

So, we sketch $y = f'(x)$:



$\rightsquigarrow$ looks like... $y = \cos x$

Indeed, $(\sin x)' = \cos x.$

Lecture 14.

Sec. 3.1 and 3.2 contain material of the pre-req. high-school course $\Rightarrow$ AT HOME REFRESHING

Sec. 3.3. Derivatives of TRIGS.

$$\boxed{\begin{array}{lll} c' = 0 & (e^x)' = e^x & \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \\ (x^n)' = n \cdot x^{n-1} & (f \pm g)' = f' \pm g' & (f \cdot g) = f'g + f \cdot g' \end{array}}$$

Last time: Intuition suggested us $(\sin x)' = \cos x$

Now, show this using first principles, i.e. „definition".

Ex. $$\sin'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cdot \cosh + \sinh \cdot \cos x - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x (\cos h - 1)}{h} + \lim_{h\to 0} \frac{\sin h \cdot \cos x}{h} =$$

does not depend on h $\Rightarrow$ can be treated as a constant

$$= \sin x \cdot \underbrace{\boxed{\lim_{h\to 0} \frac{\cos h - 1}{h}}}_{=0} + \cos x \cdot \underbrace{\boxed{\lim_{h \to 0} \frac{\sin h}{h}}}_{=1} = \cos x$$

THM: $\sin'(x) = \cos x$, $\cos'(x) = -\sin x$

Ex. $$\tan'(x) = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\overbrace{\cos^2 x + \sin^2 x}^{=1}}{\cos^2 x} = \sec^2 x$$

THM: $\tan'(x) = \sec^2 x$, $\cot'(x) = -\csc^2(x)$

Ex. $(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{1'\cdot\cos x - 1\cdot\cos' x}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \cdot \tan x$

Ex. $(\csc x)' = \left(\frac{1}{\sin x}\right)' = \frac{1'\sin x - 1\cdot \sin' x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\csc x \cdot \cot x$

Lecture 15. Derivatives of Trigs.

Ex. $\cos^{(35)}(x) = ?$

$$\left.\begin{aligned}
\cos^{(1)} x &= -\sin x\\
\cos^{(2)} x &= -\cos x\\
\cos^{(3)} x &= \sin x\\
\cos^{(4)} x &= \cos x
\end{aligned}\right\} 4$$

$$\left.\begin{aligned}
\cos^{(5)} x &= -\sin x\\
\cos^{(6)} x &= -\cos x\\
&\ldots
\end{aligned}\right] 4$$

Method 1: $\cos^{(n)} x = \cos x$ whenever n is divisible by 4.

$\Rightarrow$ take the closest number to 35 $\longrightarrow 32 = 8\cdot 4$.

$\Rightarrow \cos^{(32)} x = \cos x \Rightarrow \cos^{(33)} = -\sin x \Rightarrow$

$\Rightarrow \cos^{(34)} x = -\cos x \Rightarrow \cos^{(35)} = \sin x$

Method 2: all depends on the remainder of the division of 35 by 4:

if rem $=1 \Rightarrow -\sin x$

if rem $=2 \Rightarrow -\cos x$ $\qquad \Rightarrow 35 : 4 = 8$ rem. $3 \Rightarrow$

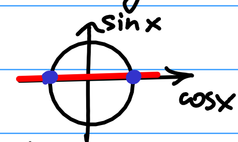
if rem $=3 \Rightarrow \sin x$ $\qquad \cos^{(35)}(x) = \sin x$

if rem $=0 \Rightarrow \cos x$

Ex. For which values x does $y = 6\cos x - 3x$ has tangent with slope -3?

$y' = -6\sin x - 3 = -3$

$\sin x = 0 \rightarrow$

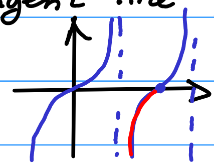


$\Rightarrow x = 0 + 2\pi k$ or $x = \pi + 2\pi k$ where $k = 0, \pm 1, \pm 2, \ldots$

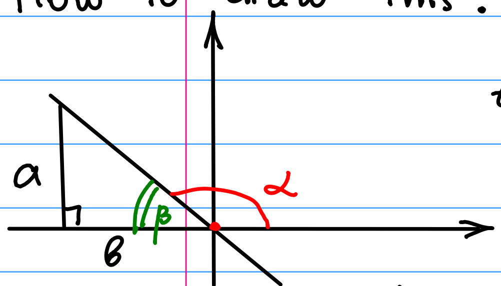
What does it mean? 1) $y'\left(\text{these } x\right) = -3 \Rightarrow$ slope of tang. line is $-3 \Rightarrow \tan\alpha = -3$

2) α – angle between Ox and tangent line, $\alpha \in [0, \pi]$

$\tan\alpha = -3 < 0 \Rightarrow \alpha \in \left[\frac{\pi}{2}, \pi\right]$

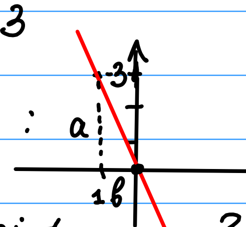


How to draw this?



$\tan(\beta) = \tan(\pi - \alpha) = -\tan\alpha = -(-3) = 3$

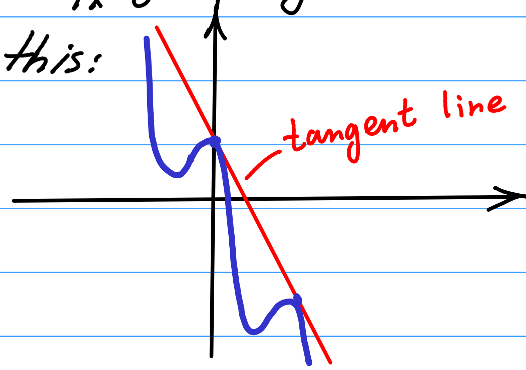
we may construct β by taking $a = 3$, $b = 1$:



What is the equation of the tangent line at one of those points, $x = 0$?

$y = -3\cdot x + b$, at $x = 0$ $\left.(6\cos x - 3x)\right|_{x=0} = 6 \Rightarrow y = -3x + 6$

If you are curious, the graph looks like this:



Derivatives in Physics

Sec. 2.7.
Sec. 3.3.

Rate of Change: $y = f(x)$ - given

Instanteneous rate of change of x w.r.t. y:
„how f changes when x changes by a very small number": $\lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$
$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

Velocity: change of position $x(t)$ as t changes a bit $\Rightarrow v(t) = x'(t)$

Acceleration: how speed $v(t)$ changes when t changes a bit $\Rightarrow x''(t)$

Ex. Let movement of an object is descr. by $P(t) = \frac{\sin(t)}{3 - 2\cos(t)}$

Find vel. and accel.

$$v(w) = P'(t) = \frac{\cos t \cdot (3 - 2\cos t) - \sin t \cdot (3 - 2\cos t)'}{(3 - 2\cos(t))^2} = \frac{3\cos t - 2\cos^2 t - 2\sin^2 t}{(3 - 2\cos(t))^2} =$$
$$= \frac{3\cos t - 2}{(3 - 2\cos(t))^2}$$

Memorize:

$(\sin x)' = \cos x$	$(\tan x)' = \sec^2 x$	$(\sec x)' = \sec x \cdot \tan x$
$(\cos x)' = -\sin x$	$(\cot x)' = -\csc^2 x$	$(\csc x)' = -\csc x \cdot \cot x$

Chain Rule. Sec. 3.4.

$$(\sin(2x))' = ? \qquad (\sqrt{5x+1})' = ? \qquad (e^{x^2})' = ?$$

Such derivatives can be computed using

THM. If $F(x) = f \circ g(x) = f(g(x))$, $g(x)$ - diff. at x, f - diff. at $g(x)$

$$\Rightarrow \quad F'(x) = f'(g(x)) \cdot g'(x)$$

Or, in Leibniz notation: $\frac{dF}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$, $u = g(x)$

Informally: $F'(x) = (\text{outside})' \cdot (\text{inside})'$

Ex. $(\sin 2x)' = ?$ $\quad f(x) = \sin x \quad g(x) = 2x$, $u = g(x) = 2x$

$$(\sin 2x)' = \frac{d\sin u}{du} \cdot \frac{du}{dx} = \cos u \cdot \frac{d(2x)}{dx} = \cos(2x) \cdot 2 = 2\cos 2x$$

Ex. $(\sqrt{5x+1})' = ?$ $\quad f(x) = \sqrt{x}$, $g(x) = 5x+1 \quad u = g(x) = 5x+1$

$$(\sqrt{5x+1})' = \frac{d\sqrt{u}}{du} \cdot \frac{d(5x+1)}{dx} = (u^{\frac{1}{2}})' \cdot 5 = \frac{1}{2} \cdot u^{-\frac{1}{2}} \cdot 5 = \frac{5}{2} \cdot \frac{1}{\sqrt{u}} =$$
$$= \frac{5}{2} \cdot \frac{1}{\sqrt{5x+1}}$$

Lecture 16.

Ex. $\left(\sec(1-5x)\right)' = \dfrac{d\sec u}{du} \cdot \dfrac{d(\overset{u}{1-5x})}{dx} = \tan u \cdot \sec u \cdot (-5) =$

Ex. $\left((2t^3+\cos t)^{50}\right)' = 50 \cdot (2t^3+\cos t)^{49} \cdot (2t^3+\cos t)' =$
$= 50 \cdot (2t^3+\cos t)^{49} \cdot (6t^2 - \sin t)$

Derivatives of e^x and a^x

$(a^x)' = ?$

$$(a^x)' = \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = \lim_{h\to 0} \frac{a^x(a^h-1)}{h} = a^x \cdot \lim_{h\to 0} \frac{a^{h+0}-a^0}{h} =$$

$= a^x \cdot (a^x)'\big|_{x=0}$ = we stuck!
Derivative is expressed via deriv.

$\Rightarrow$ HERE COMES "e"

There are several definitions of "e". They are equivalent (i.e. define the same)

DEFS. 1. $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

2. e is the unique positive number for which $\lim_{h\to 0} \dfrac{e^h-1}{h} = 1$

$\Rightarrow (e^x)' = e^x \cdot \lim_{h\to 0} \dfrac{e^h-1}{h} = e^x \Rightarrow$ **Thm.** $(e^x)' = e^x$.

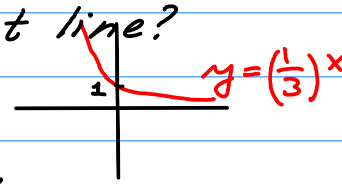
NOW $(a^x)' = ?$ $\quad a = e^{\ln a} \Rightarrow (a^x)' = \left(e^{\ln a \cdot x}\right)' = e^{\ln a \cdot x} \cdot (\ln a \cdot x)'$
$= a^x \cdot \ln a$ $\quad$ **THM.** $(a^x)' = a^x \cdot \ln a$

Ex. *It is a useful ex. to show equiv. of definitions of number e.*

$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \Rightarrow \lim_{n\to\infty} e^{1/n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)$

$h = \frac{1}{n}$, if $n\to\infty \Rightarrow h\to 0 \Rightarrow$ $\lim_{h\to 0} \dfrac{e^h-1}{h} = 1.$

Ex. At which points $y = 3^{-x}$ has horizontal tangent line?



$(3^{-x})' = \left((3^{-1})^x\right)' = (3^{-1})^x \cdot \ln(3^{-1}) = \underbrace{\left(\tfrac{1}{3}\right)^x}_{>0} \cdot \underbrace{(-\ln 3)}_{<0}$

for all x $\Rightarrow$ never 0 $\Rightarrow$ no hor. tan. lines.

What is then $y=0$? **ASYMPTOTE, not tangent line.**

Long Chains.

Ex. $\left(\sin(\cos(\sqrt{x}))\right)' = \cos(\cos(\sqrt{x})) \cdot (\cos\sqrt{x})' =$
$= \cos(\cos\sqrt{x}) \cdot (-\sin\sqrt{x}) \cdot (\sqrt{x})' = \cos(\cos\sqrt{x}) \cdot (-\sin\sqrt{x}) \cdot \left(x^{\frac{1}{2}}\right)' =$
$= \cos(\cos\sqrt{x}) \cdot (-\sin\sqrt{x}) \cdot \left(\frac{1}{2}\right) \cdot x^{-\frac{1}{2}} = -\frac{1}{2}\cos(\cos\sqrt{x}) \cdot \sin(\sqrt{x}) \cdot \frac{1}{\sqrt{x}}$

Ex. $\left(3^{\sin(x^2)}\right)' = 3^{\sin(x^2)} \cdot \ln 3 \cdot (\sin x^2)' = 3^{\sin(x^2)} \cdot \ln 3 \cdot \cos(x^2) \cdot (x^2)' =$
$= 3^{\sin(x^2)} \cdot \ln 3 \cdot \cos(x^2) \cdot 2x.$

Implicite Differentiation. Sec. 3.5.

So far we have considered only functions in the form $y = f(x)$, i.e. functions defined explicitly.

Implicitly defined functions:

Ex. $x^3 - 3xy + y^3 = 1$ **Ex.** $\sin(x+y) = y^2 \cos x$ → hard to express y in terms of x.

Ex. $x \cdot \cos y + y \cdot \cos x = 1$

How to diff. such functions?

Ex. $xy = 1$ $\frac{dy}{dx} = ?$

1) $(xy)' = (1)'$
$x \cdot y' + x' \cdot y = 0$
$x \cdot y' + y = 0$

2) Solve for y':
$y' = -\frac{y}{x}$ ← the answer.

Most answers here will involve BOTH x and y.

Remark: $xy = 1 \Rightarrow y = \frac{1}{x}$ ($x=0$ does not belong to the domain)

let's plug $y = \frac{1}{x}$ into $y' = -\frac{y}{x}$:

$$y' = -\frac{1/x}{x} = -\frac{1}{x^2} \Rightarrow \text{the same as if we compute } \left(\frac{1}{x}\right)'$$

Ex. $y^5 + y^3 = x^3 + 3$

1) $\frac{dy^5}{dx} = \frac{dy^5}{dy} \cdot \frac{dy}{dx} = 5y^4 \cdot y'$
$\frac{dy^3}{dx} = \frac{dy^3}{dy} \cdot \frac{dy}{dx} = 3y^2 \cdot y'$
$\Rightarrow 5y^4 \cdot y' + 3y^2 y' = 3x^2 + 3$

2) Solve for y': $y' = \frac{3x^2}{5y^4 + 3y^2}$

Lecture 17.

Ex. Find the slope of the tangent line at $(0,1)$ to the graph of $x \cdot \cos y + y \cdot \cos x = 1$

1) $(x \cdot \cos y)' = x' \cos y + x \cdot (\cos y)' = \cos y + x \cdot (-\sin y) \cdot y'$
$(y \cdot \cos x)' = y' \cdot \cos x + y \cdot (\cos x)' = y' \cdot \cos x + y \cdot (-\sin x)$

Together: $\cos y - x\underline{y'} \cdot \sin y + \underline{y'} \cdot \cos x - \sin x \cdot y = 0.$

2) Solve for y': $y' = \frac{y \cdot \sin x - \cos y}{\cos x - x \cdot \sin y}$

$$\text{slope } m = \tan\alpha = y'\Big|_{(0,1)} = \frac{1 \cdot \sin 0 - \cos 1}{\cos 0 - 0 \cdot \sin 1} = \frac{-\cos 1}{1} = -\cos 1$$

Derivatives of Inverse Trig. Functions.

$(\arcsin x)' = ?$

$y = \arcsin x$ means $\sin y = x$, $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

1) $(\sin y)' = \frac{d\sin y}{dy} \cdot \frac{dy}{dx} = \cos y \cdot y'$

$\Rightarrow \cos y \cdot y' = 1$

2) $y' = \frac{1}{\cos y} = \frac{1}{\cos(\arcsin x)} = ?$

$\uparrow$ nice if this were „sin"

Creating „sin" trick: $(\cos y)^2 + (\sin y)^2 = 1$

$\cos y = \pm\sqrt{1-(\sin y)^2}$

But $y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \cos y = +\sqrt{1-(\sin y)^2}$

Now: $y' = \frac{1}{\sqrt{1-(\sin(\arcsin x))^2}} = \frac{1}{\sqrt{1-x^2}}$

THM: $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ $\quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

$(\underbrace{\arctan x}_{=y})' = ?$

$\Rightarrow \tan y = x$, $(\tan y)' = x'$

$\sec^2 y \cdot y' = 1 \Rightarrow y' = \frac{1}{\sec^2 y}$

Recall:
$\sec^2 x = 1 + \tan^2 x$

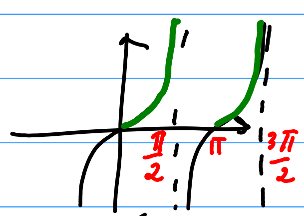
$y' = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

THM: $(\arctan x)' = \frac{1}{1+x^2}$ $\quad (\operatorname{arccot} x)' = -\frac{1}{1+x^2}$

$(\underbrace{\sec^{-1} x}_{y})' = ? \Rightarrow \sec y = x \Rightarrow \sec y = x$

$\underline{y \in [0, \pi/2) \cup [\pi, \frac{3\pi}{2})}$ $\quad \tan y \cdot \sec y \cdot y' = 1$

$y' = \frac{1}{\underbrace{\tan y}_{\shortparallel} \cdot \underbrace{\sec y}_{x}} = \frac{1}{x\sqrt{1-x^2}}$



$\tan > 0 \Longrightarrow \oplus\sqrt{1-\sec^2 x}$

Ex. $(\tan^{-1}(\sin x))' = \frac{\cos x}{1+(\sin x)^2}$

Derivatives of Inverses.

$y = f(x) \Rightarrow x = f^{-1}(y)$ where defined

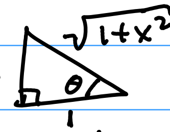
$\underbrace{x'}_{\shortparallel\, 1} = \frac{df^{-1}(y)}{dy} \cdot y' \Rightarrow y' = \frac{1}{(f^{-1}(y))'}$

THM. If $f(x)$ and $g(x)$ are inverses of each other, then
$$(f(x))' = \frac{1}{g'(f(x))}$$

Ex. $g(x) = \tan x$, $f(x) = \arctan x$

$$\Rightarrow (\arctan x)' = \frac{1}{\tan'(\arctan x)} = \frac{1}{\sec^2(\arctan x)}$$

denote: $\arctan x = \theta \Rightarrow \tan\theta = x \Rightarrow$



$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \cos\theta > 0 \Rightarrow \cos\theta = \frac{1}{\sqrt{1+x^2}}$

lecture 18.

Derivative of $\log_a x$. Sec. 3.6.

Method 1: using deriv. of inverses:

$y = \log_a x$ is inverse for $y = a^x \Rightarrow$

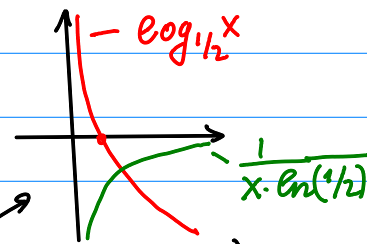
$$(\log_a x)' = \frac{1}{(a^x)'\big|_{x=\log_a x}} = \frac{1}{a^{\log_a x} \ln a} = \frac{1}{x \cdot \ln a}$$

Method 2: use implicit diff:

let $y = \log_a x \Rightarrow x = a^y \Rightarrow x' = (a^y)' \Rightarrow 1 = a^y \cdot \ln a \cdot y'$

$$\Rightarrow y' = \frac{1}{a^y \cdot \ln a} \Rightarrow y' = \frac{1}{a^{\log_a x} \ln a} = \frac{1}{x \cdot \ln a}$$

THM. $$(\log_a x)' = \frac{1}{x \cdot \ln a}$$



$$\Rightarrow (\ln x)' = (\log_e x)' = \frac{1}{x \cdot \ln e} = \frac{1}{x}, \quad (\ln x)' = \frac{1}{x}$$

1) $f' < 0$ when $f \searrow$
2) when f changes slowly $|f'|$ is small
3) when f changes rapidly, $|f'|$ is large

Ex. $R(w) = 4^w - 5\log_9 w$ $\quad \frac{dR}{dw} = 4^w \ln 4 - 5\frac{1}{w \ln 9} = 4^w 2\ln 2 - \frac{5}{2w\ln 3}$

Ex. $(\ln|x|)' = ?$

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases} \Rightarrow \ln|x| = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

$$\Rightarrow y' = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{-x} \cdot (-1), & x < 0 \end{cases} \Rightarrow$$

THM. $$(\ln|x|)' = \frac{1}{x}$$

Ex. $$\left(\ln|x+3|^{1/5}\right)' \overset{\text{Q}}{=} \left(\frac{1}{5} \cdot \ln|x+3|\right)' = \frac{1}{5(x+3)}$$

Logarithmic Differentiation.

Ex. $y = \dfrac{x^5}{(1-10x)\cdot\sqrt{x^2+2}}$ $\qquad y' = \dfrac{5x^4\cdot \text{determ.} - x^5\cdot(\text{determ})'}{(1-10x)^2\cdot(x^2+2)}$ ← too long!

1) Take ln of Both sides: $\ln y = 5\ln x - \ln(1-10x) - \frac{1}{2}\ln(x^2+2)$

2) implicite diff: $\frac{1}{y}\cdot y' = 5\cdot\frac{1}{x} - \frac{-10}{1-10x} - \frac{1}{2}\cdot\frac{2x}{x^2+2}$

$$y' = \left(\frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2}\right)\cdot\frac{x^5}{(1-10x)\cdot\sqrt{x^2+2}}$$

Ex. prove $(x^n)' = nx^{n-1}$: $\quad y = x^n$

$\ln y = n\ln x \Rightarrow \frac{1}{y}y' = \frac{n}{x}$

$\Rightarrow y' = \frac{n}{x}\cdot y = n\cdot x^{n-1}$

Lecture 19.

Ex. $y = x^x$ $\qquad y' = ?$

We know: $\quad (a^x)' = a^x\ln a \quad$ and $\quad (x^n)' = nx^{n-1}$

NEITHER WORKS, because we need BOTH the base and the exponent to contain a variable.

Method of log. diff-n: $\ln y = x\ln x \Rightarrow \frac{y'}{y} = x\cdot(\ln x)' + x'\cdot\ln x$

$y' = y\cdot(1+\ln x)$

$y' = x^x\cdot(1+\ln x)$

Ex. $y = (1-3x)^{\cos x}$

1) $\ln y = \cos x\ \ln(1-3x)$

2) $\frac{y'}{y} = -\sin x\cdot\ln(1-3x) + \cos x\cdot\frac{-3}{1-3x}$ $\qquad$ 3) $y' = \ldots$

Ex. $\left(\left(\frac{3}{4}\right)^{\cos x}\right)' = (\cos x)'\cdot\left(\frac{3}{4}\right)^{\cos x}\cdot\ln\frac{3}{4} = -\sin x\cdot\left(\frac{3}{4}\right)^{\cos x}\cdot\ln\frac{3}{4}$

THM. 1) $\left(f(x)^b\right)' = b\,f(x)^{b-1}$ $\qquad$ 3) $\left(f(x)^{g(x)}\right)'$ – we impl. diff.

2) $\left(b^{g(x)}\right)' = b^{g(x)}\ln b\cdot g'(x)$

Lecture 20.

Famous limits connected with e.

THM. $\lim\limits_{h\to 0} \dfrac{e^h - 1}{h} = 1$

Proof. recall that „e" has been chosen so that the slope at $x=0$ is $1 \Rightarrow$

$$1 = m = f'(0) = \lim_{h\to 0} \frac{e^h - e^0}{h} = \lim_{h\to 0}\frac{e^h-1}{h} \quad \text{Q.E.D.}$$

Notice that we can re-write $\lim\limits_{h\to 0}\dfrac{e^h-1}{h} = 1$ as $\lim\limits_{h\to 0}(e^h - 1) = \lim\limits_{h\to 0} h$

$\Rightarrow \lim\limits_{h\to 0} e^h = \lim\limits_{h\to 0}(h+1) \Rightarrow \lim\limits_{h\to 0}(e^h)^{1/h} = \lim\limits_{h\to 0}(h+1)^{1/h} \Rightarrow$ we proved a thm:

THM. $\lim\limits_{h\to 0}(h+1)^{1/h} = e$

Denote $n = \frac{1}{h}$ then when $h\to 0$, $n \to \infty$, then this limit can be re-written as

THM. $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

Examples on Differentiation.

Ex. $\dfrac{d}{dx}\arccos(3^x) = -\dfrac{1}{\sqrt{1-(3^x)^2}} \cdot (3^x)' = -\dfrac{3^x \ln 3}{\sqrt{1-3^{2x}}}$

Ex. Find an equation of the tangent line to the graph of $\sin(x+xy) = x \cdot y^2$ at $(\pi, 0)$.

Solution: 1) $y = m\cdot x + b$, where $m = \dfrac{dy}{dx}$ at $(\pi, 0)$

2) the function is given implicitly $\Rightarrow$ use implicit diff-on:

$$\cos(x+y)\cdot(x+xy)' = (xy^2)'$$

$$\cos(x+y)\cdot(x' + xy' + x'y) = x'y^2 + x'(y^2)'$$

$$\cos(x+y)\cdot(1 + xy' + y) = y^2 + 2y\cdot y'$$

Here we would normally express y' in terms of others, but we need only y' at $(\pi, 0) \Rightarrow$ substitute immediately:

$$\cos(\pi + 0)\cdot(1 + \pi\cdot y' + 0) = 0^2 + 2\cdot 0\cdot y'$$

$$-(1+\pi y') = 0$$

$$y' = -\frac{1}{\pi}$$

3) Equation is, therefore, $y = -\frac{1}{\pi}x + b$

the tangent line must pass through $(\pi, 0) \Rightarrow$

$$0 = -\frac{1}{\pi}\cdot\pi + b \Rightarrow b = 1$$

Answer: $y = -\frac{1}{\pi}x + 1$

Ex. Find $\frac{dy}{dx}$ if $y = \arcsin^3(2x)$

Solution: $\frac{dy}{dx} = y' = \left(\left(\arcsin(2x)\right)^3\right)' = 3\left(\arcsin(2x)\right)^2 \cdot \left(\arcsin 2x\right)'$

$$y' = 3\left(\arcsin(2x)\right)^2 \cdot \frac{1}{\sqrt{1-(2x)^2}} \cdot (2x)'$$

$$y' = 3\left(\arcsin(2x)\right)^2 \cdot \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$$

Ex. Find $\frac{dy}{dx}$ if $y = 3^{\log_{10} x}$

Solution: $\frac{dy}{dx} = y' = 3^{\log_{10} x} \cdot \ln 3 \cdot (\log_{10} x)' =$

$$= 3^{\log_{10} x} \cdot \ln 3 \cdot \frac{1}{x \cdot \ln 10}$$

Ex. Find $\frac{dy}{dx}$ if $y = \log_3(\tan(x-y))$

Solution: Function has given implicitly → use impl. diff-on:

$$y' = \frac{1}{\ln 3 \cdot \tan(x-y)} \cdot (\tan(x-y))'$$

$$y' = \frac{1}{\ln 3 \cdot \tan(x-y)} \cdot \sec^2(x-y) \cdot (x-y)'$$

$$y' = \frac{1}{\ln 3 \cdot \tan(x-y)} \cdot \sec^2(x-y) \cdot (x' - y')$$

$$y' = \frac{1}{\ln 3 \cdot \tan(x-y)} \cdot \sec^2(x-y) \cdot (1 - y')$$

Express y'!

$$y'\left(1 + \frac{1}{\ln 3 \cdot \tan(x-y)} \cdot \sec^2(x-y)\right) = \frac{1}{\ln 3 \cdot \tan(x-y)} \cdot \sec^2(x-y)$$

$$y' = \frac{\frac{1}{\ln 3 \cdot \tan(x-y)} \cdot \sec^2(x-y)}{1 + \frac{1}{\ln 3 \cdot \tan(x-y)} \cdot \sec^2(x-y)}$$

Ex. y' if $y = \log_5(x \cdot 5^x)$

Solution: $\log_5(x \cdot 5^x) = \log_5 x + \log_5 5^x = \log_5 x + x$

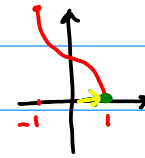
$$y' = \frac{1}{\ln 5 \cdot x} + 1$$

Lecture 21. Review.

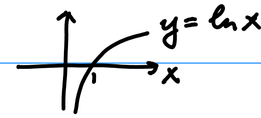
Ex. $\lim\limits_{x\to 0} \ln\left(\arccos(1-x^2)\right) =$

1) $x\to 0$
$x^2 \to 0+$
$-x^2 \to 0-$
$1-x^2 \to 1-$

2) $\arccos(\overset{\nearrow 1-}{1-x^2}) \to 0+$



3) $\ln(\to 0+) \longrightarrow = -\infty$



Ex. $\lim\limits_{x\to 0} x^2\cdot\cos\left(\frac{1}{x}\right) = ?$ TYPE: „$0\cdot$ DNE"

$-1 \le \cos\left(\frac{1}{x}\right) \le 1$

$-x^2 \le x^2\cdot\cos\left(\frac{1}{x}\right) \le x^2$ $\Rightarrow$ by Squeeze thm. $\lim\limits_{x\to 0} x^2\cdot\cos\frac{1}{x} =$

$x\to 0$: $-x^2 \to 0$, $x^2 \xrightarrow{x\to 0} 0$

Ex. $\lim\limits_{x\to\infty} \frac{\sin 5x}{5x} = ?$ Type „$\frac{DNE}{\infty}$"

$-1 \le \sin 5x \le 1$

$-\frac{1}{5x} \le \frac{\sin x}{5x} \le \frac{1}{5x}$ $\Rightarrow$ $\lim\limits_{x\to\infty} \frac{\sin x}{x} = 0.$

$x\to\infty$: $-\frac{1}{5x} \to 0$, $\frac{1}{5x} \xrightarrow{x\to\infty} 0$

Ex. $\lim\limits_{x\to 4-} \frac{x^2-4x}{x^2-8x+16} = \lim\limits_{x\to 4-} \frac{x(x-4)}{(x-4)^2} = \lim\limits_{x\to 4-} \frac{x}{x-4}$ TYPE „$\frac{const}{0}$" $= \pm\infty$

$\Rightarrow$ figure out the sign: $x \to 4-$ „+"; $x-4 \xrightarrow[x\to 4-]{}$ „−" $\Rightarrow$ $-\infty$

Ex. $\lim\limits_{x\to 4} \frac{4-t}{2-\sqrt{t}} = \lim\limits_{x\to 4} \frac{\cancel{(2-\sqrt{t})}\cdot(2+\sqrt{t})}{\cancel{2-\sqrt{t}}} = 4.$

Ex. $\left(x^{\frac{1}{4}}\right)' = \lim\limits_{h\to 0} \frac{(x+h)^{\frac{1}{4}} - x^{1/4}}{h} = \lim\limits_{h\to 0} \frac{\left((x+h)^{\frac{1}{4}} - x^{1/4}\right)\cdot(\ldots + \ldots)}{h\cdot(\ldots + \ldots)} =$

$$= \lim_{h\to 0} \frac{(x+h)^{\frac{1}{2}} - x^{\frac{1}{2}}}{h\cdot\left((x+h)^{\frac{1}{4}} + x^{1/4}\right)}$$

$$= \lim_{h\to 0} \frac{\left((x+h)^{\frac{1}{2}} - x^{\frac{1}{2}}\right)\cdot\left((x+h)^{\frac{1}{2}} + x^{1/2}\right)}{h\cdot\left((x+h)^{\frac{1}{4}} + x^{1/4}\right)\cdot\left((x+h)^{\frac{1}{2}} + x^{1/2}\right)} =$$

$$= \lim_{h\to 0} \frac{x+h-x}{h\cdot\left((x+h)^{\frac{1}{4}} - x^{1/4}\right)\cdot\left((x+h)^{\frac{1}{2}} + x^{1/2}\right)} = \frac{1}{2x^{1/4}\cdot 2x^{1/2}} = \frac{1}{4x^{3/4}}$$

Lecture 23.
Review 3.

2 Types of questions:

1) trig (inverse trigs) = ? $\longrightarrow$ △
2) inverse trig (trig) = ? $\longrightarrow$ cancellation rules

But mind domains!

Cancellation Rules:

$\cos(\arccos x) = x$ for all x in the domain

$\arccos(\cos x) = x$ for $0 \le x \le \pi$.

$\sin(\arcsin x) = x$ for all x in the domain

$\arcsin(\sin x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Useful: $\arcsin(-x) = -\arcsin(x)$ ← because $\arcsin x$ odd

$\arccos(-x) = \pi - \arccos(x)$ ← $\cos(\arccos(-x)) = \cos(\pi - \arccos x)$

$-x = \cos\pi \cdot \cos(\arccos x) + \sin\pi \cdot \sin(\arcsin x)$

$-x = -x$ - true.

Ex. $\cos(\arccos(-\frac{1}{3})) = -\frac{1}{3}$

Ex. $\arccos(\cos\frac{7\pi}{6}) = ?$ $\quad \frac{7\pi}{6} > \pi \Rightarrow$ cannot apply

$\cos(\frac{7\pi}{6}) = \cos(\frac{\pi}{6} + \pi) = \cos\frac{\pi}{6} \cdot \underbrace{\cos\pi}_{=-1} - \sin\frac{\pi}{6} \cdot \underbrace{\sin\pi}_{=0} = -\frac{\sqrt{3}}{2}$

$\arccos(-\frac{\sqrt{3}}{2}) = \pi - \arccos(\frac{\sqrt{3}}{2}) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Alternatively: $\arccos(-\frac{\sqrt{3}}{2}) = u$

$\cos u = -\frac{\sqrt{3}}{2}$ and $u \in [0, \pi]$ $\Rightarrow u = \frac{5\pi}{6}$

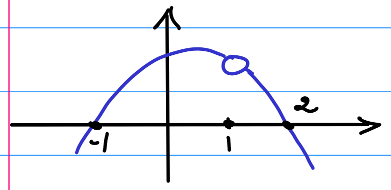
Ex. $\arccos(\cos 2) = 2$ because $0 \le 2 \le \pi$

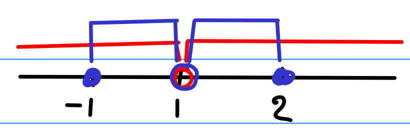
Ex. $\tan(\arccos(-\frac{\sqrt{2}}{2})) = ?$

$\arccos(-\frac{\sqrt{2}}{2}) = \pi - \arccos(\frac{\sqrt{2}}{2}) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\tan(\frac{3\pi}{4}) = \tan(\pi - \frac{3\pi}{4}) = -\tan\frac{\pi}{4} = -1$

Ex. Find x for which $y = \sqrt{f(x)}$ continious.



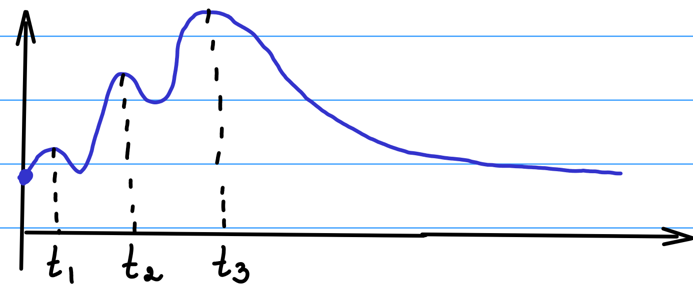
1) $f(x)$ cont. on $(-\infty; 1] \cup [1, +\infty)$
2) we need to plug $f(x)$ under $\sqrt{f(x)}$ $\Rightarrow f(x)$ must be $\ge 0 \iff$ $x \in [-1, 1) \cup (1, 2]$
3) 

$\Rightarrow x \in [-1, 1) \cup (1, 2]$

Lecture 24. (material for „after" midterm)

Sec. 4.1. Minimum and Maximum Values.

Suppose that the expected price of shares can be described by:



When it is a good time to sell them?

Def. Let $f(x)$ be a function.

$f(c)$ is its LOCAL MIN if   for All x from some $(\;\underset{c}{\cdot}\;)$ and $x \in$ Domain $f(c) \le f(x)$

$f(c)$ is its LOCAL MAX if $f(c) \ge f(x)$

$f(c)$ is its GLOBAL MIN if $f(c) \le f(x)$ for all x

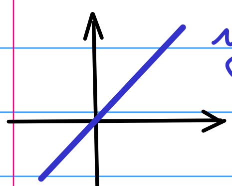
$f(c)$ is its GLOBAL MAX if $f(c) \ge f(x)$ from the domain

(absolute)

Def. All max and min — „extreme values" of $f(x)$

Q: Does EVERY function has min and max?

Ex.

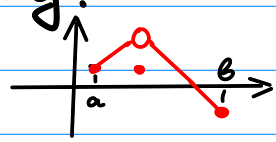


$\rightarrow$ no as $y=x$ increases without bound as $x \to \infty$ and decreases without bound as $x \to -\infty$

Extremuses may NOT exist even if we restrict ourselves on some interval

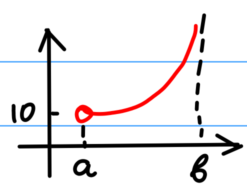
What can go wrong?

Discontinuity:



no max on $[a,b]$

Open interval: if $f(x)$ is defined on an open interval.



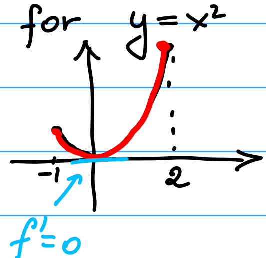
no max as $f(x) \longrightarrow \infty$ on the right

no min as $f(x) \to 10$ on the left never reaching it

Nothing can go wrong with ...

THE EXTREME VALUE THM. If $f(x)$ is cont. on $[a,b]$ then $f(x)$ has **abs.** max and min on $[a,b]$.

Ex. for $y=x^2$ on $[-1,2]$



local min: at $x=0$,

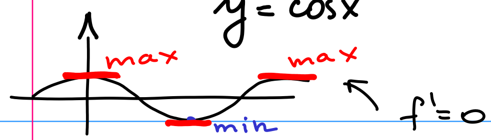
local max: at $x=-1$, $x=2$

abs. min: at $x=0$,

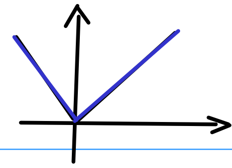
abs. max: at $x=2$,

When local or abs. max and min may appear?

Ex. $y = \cos x$



Ex.



$y = |x|$
no derivative at $x=0$,
But min is there!

Def. Point c from domain of $f(x)$ is called **CRITICAL** if $f'(c) = 0$ OR $f'(c)$ does not exist.

FERMAT'S THM: if c is local min or max for $f(x) \Rightarrow$ c is its critical point.

Ex. extr. values for $y = \cos x$

critical points: 1) $y' = -\sin x = 0 \Rightarrow x = \pi k,\ k \in \mathbb{Z}$

2) y' exists everywhere

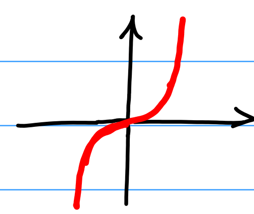
Ex. $y = |x|$

$$y = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases} \qquad y' = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

1) derivative exists everywhere except at $x=0$ $\Rightarrow$ critical point $x=0$ only.
2) derivative is nowhere $= 0$

Ex.

$y = x^3$
$y' = 3x^2$, $\quad y'(x) = 0 \iff x = 0$
But there is **NO** extr. **at all.**
Only horiz. tangent line is there!



Not every critical point is extremum!!

THM. If $f(x)$ is cont. on CLOSED interval $[a, b]$ then if extremum exists AT ALL, it is either in a critical point or in $x = a$ or in $x = b$.

Lecture 25.

Finding Absolute extrema of $f(x)$ on $[a,b]$

Step 1. Verify that function is continious on the interval $[a,b]$

Step 2. Make a list of testing points:

1) f' DNE
2) $f'=0$
3) dubious points

} all critical points are here

take only those which belong to $[a,b]$

Step 3. Evaluate $f(x)$ at these points.

Step 4. Compare these values. $\Longrightarrow$ The largest – Abs. max. the smallest – Abs. min.

Simple Ex. $g(t) = 2t^3+3t^2-12t+4$ on $[-4, 2]$

1. $g(t)$ is cont. on $[-4,2]$
2. TESTING POINTS: $g'(t)$ exists everywhere $\Rightarrow$ no „dubious" points

   $g'(t) = 6t^2+6t-12 = 6\cdot(t+2)\cdot(t-1) = 0 \Longleftrightarrow$ $t=-2$ and $t=1$
3. $g(-2) = 24 \quad g(1) = -3 \quad g(-4) = -28 \quad g(2) = 8$
4. $24 > 8 > -3 > -28$

   Therefore, 24 – abs. max and -28 – abs. min.

Ex. $g(t) = 2t^3+3t^2-12t+4$ on $[0, 2]$

almost the same as above, but the interval is different.

1. the same as above
2. the same as above: $x=-2$ and $x=1$, but only $x=1 \in [0,2]$

   $\Rightarrow$ critical point to consider: $x=1$ only
3. $g(0) = 4 \quad g(2) = 8 \quad g(1) = -3$
4. $8 > 4 > -3 \Rightarrow$ abs. min is -3 and occurs at $x=1$

   abs. max is 8 and occurs at $x=2$

$\Rightarrow$ Remember to exclude critical points not belonging to the interval.

Otherwise we would get $g(-2) = 24$ and abs. max would be found wrongly.

Ex. Suppose that the amount of money in a bank account after t years is given by $A(t) = 2000 - 10 \cdot t \cdot e^{5 - t^2/8}$
Determine the min and max amount of money in the account during the first 10 years that it is open.

Modelling: [= transforming to the language of math]
all we need to do is to determine abs. min and max for $A(t)$ on $[0, 10]$

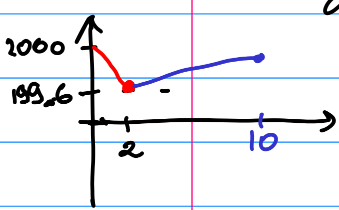
Step 1. $A(t)$ is cont. for all $t \in [0, 10]$

Step 2. critical points:

$$\frac{dA(t)}{dt} = -10 \cdot t \cdot \left(e^{5-\frac{t^2}{8}}\right)' + ((-10) \cdot t)' \cdot e^{5-\frac{t^2}{8}} =$$

$$= -10 \cdot t \cdot \left(-\frac{t}{4}\right) \cdot e^{5-\frac{t^2}{8}} - 10 \cdot e^{5-\frac{t^2}{8}}$$

$$= -10 \cdot e^{5-\frac{t^2}{8}} \left(-\frac{t^2}{4} + 1\right) = 0 \iff t = \pm 2$$



only $t = 2$ belongs to $[0, 10]$

Step 3. $A(2) = 2000 - 20 \cdot e^{9/2} \approx 199.66$

$A(0) = 2000$

$A(10) = 2000 - 100 \cdot e^{\frac{40}{8} - \frac{100}{8}} = 2000 - 100 \cdot e^{\frac{15}{2}} \approx 1999.99$

Step 4. so the max is 2000 and it happens initially, at $t = 0$
the min will be ≈ 199.66 which occurs at the 2 years mark.

Ex. Determine the absolute extrema for $Q(y) = 3y \cdot (y+4)^{2/3}$ on $[-5, -1]$.
$-4+h$

Step 1. $Q(y)$ is cont. on $[-5, -1]$

Step 2. $Q' = (3y)' \cdot (y+4)^{2/3} + 3y \cdot \left((y+4)^{\frac{2}{3}}\right)' =$

$$= 3 \cdot (y+4)^{2/3} + 3y \cdot \left(\frac{2}{3}\right) \cdot (y+4)^{-\frac{1}{3}} = \frac{3(y+4) + 2y}{(y+4)^{1/3}} = \frac{5y+12}{(y+4)^{1/3}}$$

... to be continued tomorrow.

Advanced Homework:

$$y = x + \cos\left(x^{2/3}\right)$$

Test whether $x = 0$ is a critical point or not.

Lecture 26.

Ex. Determine the absolute extrema for $Q(y) = 3y\cdot(y+4)^{2/3}$ on $[-5,-1]$.

$-4+h$

Step 1. $Q(y)$ is cont. on $[-5,-1]$

Step 2. $Q' = (3y)'\cdot(y+4)^{2/3} + 3y\cdot\left((y+4)^{\frac{2}{3}}\right)' =$

$$= 3\cdot(y+4)^{2/3} + 3y\cdot\left(\tfrac{2}{3}\right)\cdot(y+4)^{-\frac{1}{3}} = \frac{3(y+4)+2y}{(y+4)^{1/3}} = \frac{5y+12}{(y+4)^{1/3}}$$

what happens at $y=-4$?

Step 3. $$Q'(-4) = \lim_{h\to 0} \frac{3\cdot(-4+h)\cdot h^{2/3} - 3\cdot(-4)\cdot 0}{h} =$$

$$= \lim_{h\to 0} \frac{3\cdot(h-4)\cdot}{h^{1/3}} = 3\cdot\lim_{h\to 0}\Big(\underbrace{h^{2/3}}_{\to 0} - \underbrace{\frac{4}{h^{1/3}}}_{\to\infty}\Big) = -\infty$$

DNE

$\Rightarrow$ a critical point

BUT: you can add $x=-4$ to the list of testing points as "dubious" one, without computations

$Q'(y)=0 \iff y = -\frac{12}{5} \to$ another critical point.

Step 4: $Q(-5) = -15$ $\quad Q(-1) \approx -6.241$

$Q(-4) = 0$ $\quad Q\left(-\frac{12}{5}\right) \approx -9.849$

$\Rightarrow$ min. is ≈ -9.849 which occurs at $y = -\frac{12}{5}$

max is -15 which occurs at $= -5$.

Related Rates. Sec. 3.9.

Idea: "A Rate of change in terms of other rate of change, use chain rule".

Example: Air being pumped into spherical Balloon at a rate of $5\ cm^3/min$. Determine the rate at which the radius of the Balloon is increasing when the diameter of the Ballon is $20cm$.

Modelling:

Problem: Rate, speed, slope $\iff$ Math: derivative

given: "a rate of $5\ cm^3/min$"

$cm^3/min \Rightarrow$ changing of volume with time

$\Rightarrow V'(t) = 5$

question: "rate ... radius ... when.... $d=20$"

$\Rightarrow$ find $r'(t)$ when $r(t) = \frac{d}{2} = 10$

$\Rightarrow$ **need to relate $V(t)$ and $r(t)$:**

$$V(t) = \frac{4}{3}\pi\left(r(t)\right)^3 \text{ — composite function}$$

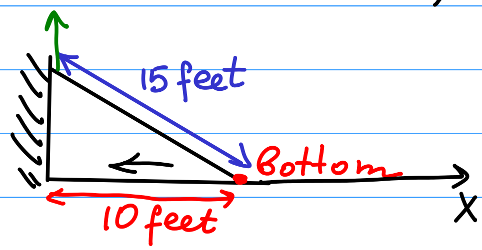
$$\frac{dV}{dt} = \frac{dV(r)}{dr}\cdot\frac{dr(t)}{dt} = \frac{4}{3}\pi\cdot 3r^2\cdot r' = 4\cdot\pi\cdot r^2\cdot r'$$

$$\Rightarrow \quad 4\pi r^2\cdot r' = 5$$

$\Rightarrow$ when $r=10$, we have $4\cdot\pi\cdot 100\cdot r' = 5 \Rightarrow r' = \frac{1}{80\pi}$ cm/min

Ex. A 15 foot ladder is resting against the wall. The Bottom is initially 10 feet away from the wall and is being pushed towards the wall at a rate of $\frac{1}{4}$ ft/sec. How fast is the top of the ladder moving up the wall 12 sec after we start pushing.

1. Sketch:



2. Model: since distance from the wall, x is decreasing then the rate is negative: $\left(x(t)\right)' = -\frac{1}{4}$

Be careful with signs in these problems!

"How fast" $\to$ speed = derivative : $\quad y'(12) = ?$

3. Find relationship between two rates: $\left(x(t)\right)^2 + \left(y(t)\right)^2 = 15^2$

Impl. diff: $\quad 2x\cdot x' + 2y\cdot y' = 0$

$\Rightarrow$ at time $t=12$ we have:

$$x(12)\cdot \underbrace{x'(12)}_{-\frac{1}{4}} + y(12)\cdot \underbrace{y'(12)}_{?} = 0$$

a) After 12 sec. x went distance $= 12\cdot$ speed $= 12\cdot\frac{1}{4} = 3$

$\Rightarrow \quad x(12) = 10 - 3 = 7$

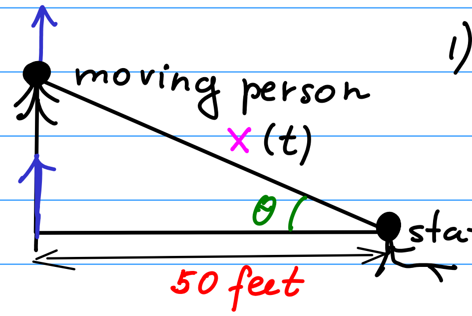
b) $\quad \underbrace{\left(x(12)\right)^2}_{49} + \left(y(12)\right)^2 = \underbrace{15^2}_{225} \Rightarrow \quad y(12) = \sqrt{176}$

$\Rightarrow \quad 7\cdot\left(-\frac{1}{4}\right) + \sqrt{176}\cdot y'(12) = 0$

$y'(12) = \ldots$

Lecture 27.

Ex. Two people are 50 feet apart. One of them starts walking north at a rate so that the angle shown in the diagram is changing at a constant rate 0.01 rad/min. At what rate is distance between two people changing when $\theta = \frac{\pi}{4}$ radians



1) Let $x = x(t)$ be distance between them

$\Rightarrow \quad \cos\theta = \frac{50}{x} \Rightarrow \sec\theta = \frac{x}{50}$

2) $\theta'(t) = 0.01$

3) We want: rate = speed = derivative, x' when $\theta = \frac{\pi}{4}$

4) diff: $(\sec\theta)' = \left(\frac{x}{50}\right)'$ w.r.t. „t"

$$\tan\theta \cdot \sec\theta \cdot \theta' = \frac{x'}{50}$$

at $\theta = \frac{1}{2}$: $\quad \tan\frac{\pi}{4} \cdot \sec\frac{\pi}{4} \cdot 0.01 = \dfrac{x'(\frac{\pi}{4})}{50}$

$$1 \cdot \sqrt{2} \cdot 0.01 \cdot 50 = x'\left(\frac{\pi}{4}\right)$$

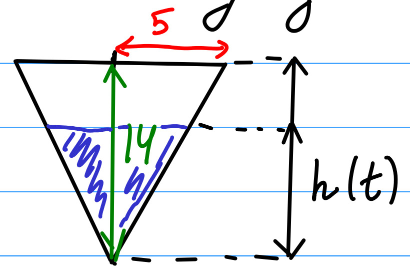
$$x'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \leftarrow \text{Answer}$$

Ex. A tank of water in a shape of cone is leaking water at a constant rate $2\ ft^3/hour$. The base radius of the tank is 5 ft and the heigh of the cone is 14 ft.

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6ft?

(b) At what rate is the radius of the top of the water changing when the depth of the water is 6ft?

1. Sketch:



2. $2\ ft^3/hour \Rightarrow V'(t) = -2$

Find: $h'(t)$ when $h(t) = 6$

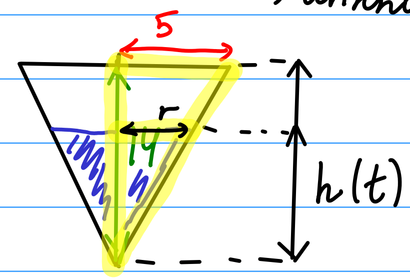
3. Relate the given derivative (rate) and the one we are searching

$$V(t) = \frac{1}{3}\cdot\pi\cdot r^2(t)\cdot h(t)$$

4. Use chain rule:

$$V' = \frac{1}{3}\pi\, 2r\cdot \underbrace{r'}_{\text{unknown}}\, h + \frac{1}{3}\pi\, r^2\cdot \underbrace{h'}_{\text{unknown}}$$

Notice two similar Δ:



$$\frac{r}{h} = \frac{5}{14}$$

$$r = \frac{5}{14}\cdot h$$

$$r' = \frac{5}{14} h'$$

5. when $h = 5$ at some time t_0, we have

$$2 = \frac{1}{3}\cdot\pi\cdot 2\cdot \underbrace{\frac{5}{14}\cdot 6}_{r} \cdot \underbrace{\frac{5}{14}\cdot h'}_{r'} \cdot \underbrace{6}_{h} + \frac{1}{3}\pi\cdot\left(\frac{5}{14}\cdot 6\right)^2\cdot h'$$

$$h' = \frac{-98}{225\pi}$$

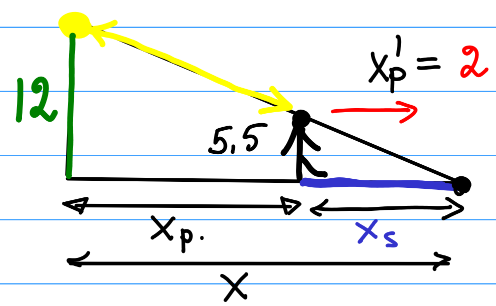
Lecture 28.

Ex. A light is on the top of a 12 ft tall pole and a 5.5 ft tall person is walking away from the pole at a rate of 2 ft/sec.

a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?

b) — " — " — — from the person — " — 25 ft.

1. Sketch:



Given: $x_p' = 2$

Find: x' when $x_p = 25$

Similar Δs:

$$\frac{12}{x} = \frac{5.5}{x_s}$$

$$12 \cdot x_s = 5.5 \cdot x$$

$$12 \cdot \underbrace{x_s'}_{\text{dont know}} = 5.5 \cdot x'$$

$$x = x_p + x_s \Rightarrow x' = \underbrace{x_p'}_{=2} + x_s' \Rightarrow x_s' = x' - 2$$

$$12(x' - 2) = 5.5 \cdot x'$$

$$6.5x' = 24 \qquad x' = \frac{24}{6.5}$$

But why did not we use $x_p = 25$?

Because it turned out that x' is constant! $\Rightarrow$ **RARE**

b) $x_s' = ?$ when $x_p = 25$

$$\underbrace{x'}_{\frac{24}{6.5}} = \underbrace{x_p'}_{2} + x_s' \Rightarrow x_s' = \ldots$$

Shape of Graph. Sec. 4.3.

THM. If $f'(x) > 0$ on an interval $\Rightarrow f(x) \nearrow$.

If $f'(x) < 0$ .... $\Rightarrow f(x) \searrow$

If $f(x) = 0$ ..... $\Rightarrow f(x)$ — „constant"

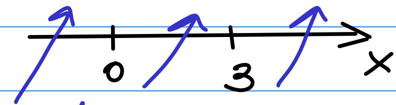
Ex. Determine all the intervals where func. ↗ or ↘

$y = -x^4 + 4x^3$

$y' = -4x^3 + 4\cdot 3\cdot x^2 = -4x^2(x-3)$ — continious function

$\Rightarrow$ the only way y' can change the sign is to go though zero.

$y' = 0 \iff \begin{matrix} x=0 \\ x=3 \end{matrix} \Rightarrow$



pick ANY point from each interval and check whether $y' > 0$ or < 0.

pick $x=-1$ for $(-\infty, 0) \Rightarrow y'(-1) = -4\cdot 1^2\cdot(-1-3) > 0$

pick $x = 1$ for $(0,3) \Rightarrow y'(1) = -4\cdot 1^2\cdot(1-3) > 0$

pick $x = 4$ for $(3, +\infty) \Rightarrow y'(4) = -4\cdot 4^2\cdot(4-3) < 0$

f: ↗ ↗ ↘

f': $+$ 0 $+$ 3 $-$ x

Answer: increase: $(-\infty, 0) \cup (0, 3)$

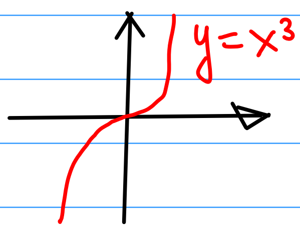
decrease: $(3, +\infty)$

Remark: A non-continuous f' may change the sign also when f' DNE

Recall:

FERMAT'S THM. If f has a local (or abs.) max or min at $x=c$ then c is a critical point.

But Recall



$y' = 3x^2 = 0 \iff x=0$ – critical

but $x=0$ is neither max nor min.

$\Rightarrow$ Not every critical point is max or min.

THE 1st DERIV. TEST Let $f(x)$ be cont.
Let c be its critical point.

Then: $f(c)$ is local max if f' changes from „−" to „+" at c.

$f(c)$ is local min if f' changes from „+" to „−" at c.

$f(c)$ has neigher if f' does NOT change the sign

Ex. Find **ABSOLUTE** extrema for $y = |x^2 - 3|$ on $[-5, 5]$

As we learned on Friday (Sec. 4.1):

Step 1 $\quad y' = \begin{cases} 2x, & x^2 \geq 3 \\ -2x, & x^2 \leq 3 \end{cases}$

Step 2: Testing points:

1) $y' = 0 \Rightarrow x = 0$ $\qquad$ 2) y' DNE $\quad x = \pm\sqrt{3}$

3) End points: $x = -5$ and $x = 5$

Step 3

$y(0) = |0-3| = 3$

$y(\pm\sqrt{3}) = |3-3| = 0$

$y(\pm 5) = |25-3| = 22$

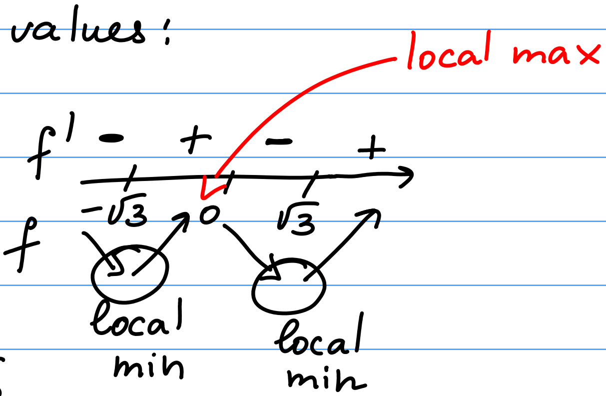
$\Rightarrow$ **ABSOLUTE** max: 22

min: 0.

Ex. Find all **LOCAL** extrema for $y = |x^2 - 3|$ on $[-5, 5]$

Step 1: Critical points: $x = 0$, $x = \pm\sqrt{3}$

Step 2: put $0, \pm\sqrt{3}$ on $\longrightarrow x$ and determine signs on the intervals by picking values:

$\Rightarrow$ 

y' has local mins: $y = 0$ at $x = \pm\sqrt{3}$

local maxs: $y = 3$ at $x = 0$

Lecture 30. Sec. 4.3.

Concavity Test: If $f''(x)>0$ on $I \Rightarrow \smile$ on I

If $f''(x)<0$ on $I \Rightarrow \frown$ on I

Inflection Point: $x=c$ for cont. function $f(x)$ if concavity changes at $x=c$.

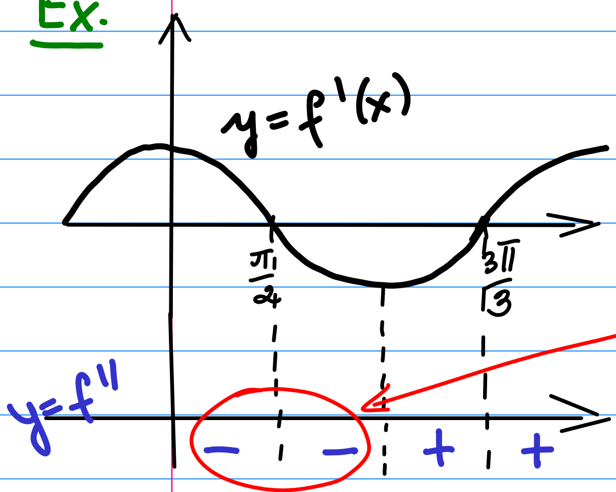
Second Derivative Test: If $f'(c)=0$, $f''(x)$ — cont. on $(\underset{c}{+})$

$f''(c)>0 \Rightarrow$ LOCAL MIN. $\smile$

$f''(c)<0 \Rightarrow$ LOCAL MAX. $\frown$

$f''(c)=0 \Rightarrow$ no conclusion.

Ex.



$f'(\frac{\pi}{2})=0$

Q: Does $x=\frac{\pi}{2}$ is the point of inflection?

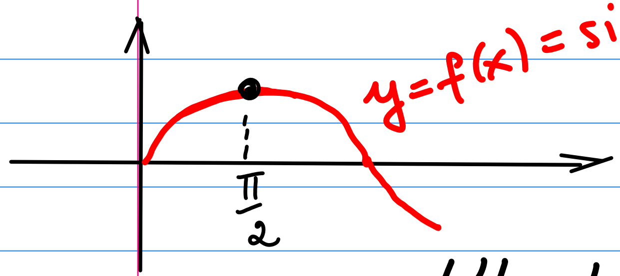
no!

Q: Does $x=\frac{\pi}{2}$ - local extrema for $f(x)$?

f': $+$, $-$ $\rightarrow$ yes local max

f: $\nearrow \pi/2 \searrow$

In this particular case we can verify this:



L'hopital's Rule. Sec. 4.4.

THM: Let $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ OR $\frac{\pm\infty}{\pm\infty}$ then $= \lim\limits_{x\to a}\frac{f'}{g'}$

where a may be $\pm\infty$ if exists or $\pm\infty$

Ex. $\lim\limits_{x\to 0}\frac{\sin x}{x} = \left[\frac{0}{0}\right] = \lim\limits_{x\to 0}\frac{\cos x}{1} = 1$

Indeterminate Product $[0\cdot(\pm\infty)]$

Ex. $\lim\limits_{x\to-\infty} x\cdot e^x = [-\infty\cdot 0] = \lim\limits_{x\to-\infty} \dfrac{e^x}{1/x} = \left[\dfrac{0}{0}\right] =$

$= \lim\limits_{x\to-\infty} \dfrac{e^x}{-1/x^2} = \left[\dfrac{0}{0}\right] = \lim\limits_{x\to-\infty} \dfrac{e^x}{+2/x^3} = \ldots$

$\Rightarrow$ try another function in the denom.:

$= \lim\limits_{x\to-\infty} \dfrac{x}{e^{-x}} = \left[\dfrac{-\infty}{+\infty}\right] = \lim\limits_{x\to-\infty} \dfrac{1}{-e^{-x}} = 0$

as $x\to-\infty$
$-x\to\infty$
$e^{-x}\to e^{\infty}\to\infty$

L'H can be hidden in ∞^0, 0^0, ∞^∞ $\to$ use $\ln$.

Ex. $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = [\infty^0]$

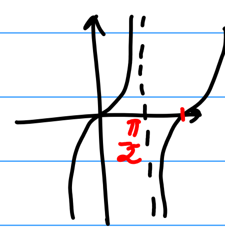
let $y = x^{\frac{1}{x}} \Rightarrow \ln \lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty} \frac{1}{x}\ln x =$

$= \lim\limits_{x\to\infty} \dfrac{\ln x}{x} = \left[\dfrac{\infty}{\infty}\right] = \lim\limits_{x\to\infty} \dfrac{1}{x} = 0$

$\Rightarrow \boxed{\lim\limits_{x\to\infty} y = 1}$

Ex. $\lim\limits_{x\to\frac{\pi}{2}^-} (\tan x)^{\cos x} = [0^0]$ let $y = (\tan x)^{\cos x}$

$\lim\limits_{x\to\frac{\pi}{2}^-} \ln y = \lim\limits_{x\to\frac{\pi}{2}^-} \overbrace{\cos x}^{0}\ \underbrace{\ln(\underbrace{\tan x}_{+\infty})}_{+\infty} = [0\cdot+\infty] =$



$= \lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{\ln(\tan x)}{\sec x} = \left[\dfrac{\infty}{\infty}\right] =$

$= \lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{\frac{1}{\tan x}\cdot\sec^2 x}{\tan x\cdot\sec x} = \lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{\sec x}{(\tan x)^2} = \lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{\frac{1}{\cos x}}{\frac{\sin^2 x}{\cos^2 x}} =$

$= \lim\limits_{x\to\frac{\pi}{2}^-} \dfrac{\cos x}{\sin^2 x} = 0 \Rightarrow \ln\left(\lim\limits_{x\to\frac{\pi}{2}^-} (\tan x)^{\cos x}\right) = 0$

$\Rightarrow$ Aswer: 1.

Optimization. Sec. 4.7.

Word problems where we are looking for a smallest or largest value that a function would take.

CLOSED I $\Rightarrow$ usual procedure, Testing points, end points.

THM. c - crit. point, f - cont., I - any interval $\Rightarrow$ if $f'>0$ for $x<c$ and $f'<0$ for $x>c$ $\Rightarrow$ $x=c$ **ABS. MAX.**

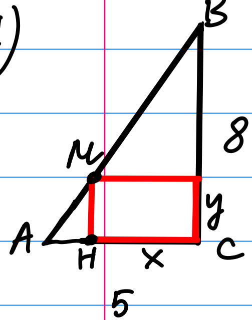
if $f'<0$ for $x<c$ and $f'>0$ for $x>c$ $\Rightarrow$ $x=c$ **ABS. MIN.**

Lecture 31.

Ex. Which dimensions of a rectangle inscribed in a right Δ with catheti 5 and 8 maximize its area?

The sides of $\square$ are parallel to the legs (= catheti) of Δ.

1)



2) $A = x \cdot y$

3) Find condition relating x and y:

similar Δ-s: $\frac{MH}{8} = \frac{5-x}{5}$ ($MH = y$)

$$y = \frac{8}{5}(5-x)$$

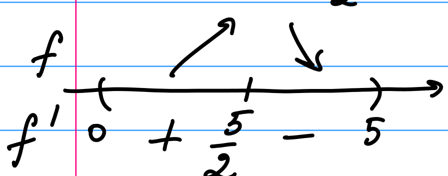
$$\Rightarrow \quad A = x \cdot \frac{8}{5} \cdot (5-x)$$

4) $A' = 8 - \frac{16}{5}x = \frac{16}{5}\left(\frac{5}{2} - x\right)$

5) Critical points: A' DNE & $A' = 0$ $\Rightarrow$ $x = \frac{5}{2}$

6) Find interval: $x \in (0,5)$ – open

7) A - cont. $x = \frac{5}{2}$ – critical $\rightarrow$ use variation of first der. test (Thm above)



for $0 < x < \frac{5}{2}$ we have $A' > 0$

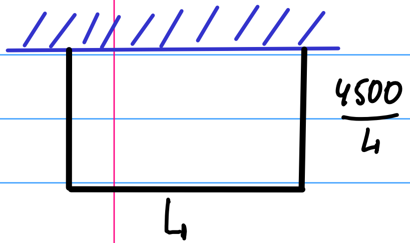
for $\frac{5}{2} < x < 5$ we have $A' < 0$

$\Rightarrow$ abs. max on $(0,5)$

Answer: $x = \frac{5}{2}$, $y = \frac{8}{5}\left(5 - \frac{5}{2}\right) = \frac{8}{5} \cdot \frac{5}{2} = 4$

$4500\ m^2$

Ex. A farmer wants to fence off a rectangular field that borders a river. The price of installing the fence at the front side is 1.6 times per m more than the other two. What should be the length of the front side so that the cost would be minimal?



$$P(L) = \text{Price}(L) = 1.6 A \cdot L + A \cdot \frac{4500}{L} \cdot 2$$

$$P'(L) = 1.6A - 2A \cdot 4500 \cdot \frac{1}{L^2} =$$

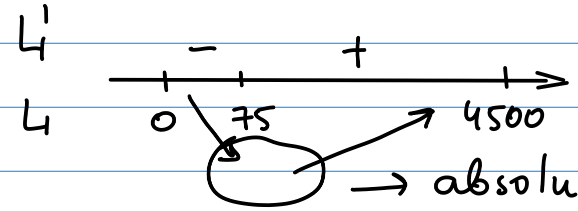
$$P'(L) = \frac{A}{10}\left(16L^2 - 20 \cdot 4500\right) / L^2$$

Critical points: $L = 0$ but $L > 0$

$$P'(L) = 0 \iff L = \sqrt{\frac{20 \cdot 45 \cdot 10^2}{16}} \quad \text{as} \quad L > 0$$

$$L = \frac{10}{4}\sqrt{4 \cdot 5 \cdot 5 \cdot 9} = \frac{\overset{5}{\cancel{10}} \cdot 5 \cdot \cancel{2} \cdot 3}{4} = 75$$

Interval: $(0, 4500)$ – open



→ absolute minimum on $(0, 4500)$

Answer: 75.

Antiderivatives. Sec. 4.9.

Def. F is antiderivative of f on I if $F' = f$ on I.

let $F' = f = G' \Rightarrow (F - G)' = 0 \Rightarrow F - G = \underbrace{C}_{\text{number}}$

$\Rightarrow$ if one knows one antiderivative, $F(x)$ all others can be written as $\underbrace{F(x) + C}_{\text{called General Antider.}}$.

Ex. G. antiderivative of $f(x) = \frac{1}{\sqrt{1-x^2}}$ is $F(x) = \arcsin x + C$

Lecture 32.

Ex. A particle is moving with $v(t) = \frac{1}{1+t^2}$. Find function $s(t)$ describing the position of the particle if $s(1) = 0$

$$s'(t) = v(t)$$
$$s(t) = \arctan t + C$$
$$s(0) = \arctan 1 + c = 0 \quad \Rightarrow c = -\frac{\pi}{4}$$
$$\Rightarrow s(t) = \arctan t - \frac{\pi}{4}$$

Sigma Notation. Appendix E.

Capital sigma Σ is used for writing the sums:

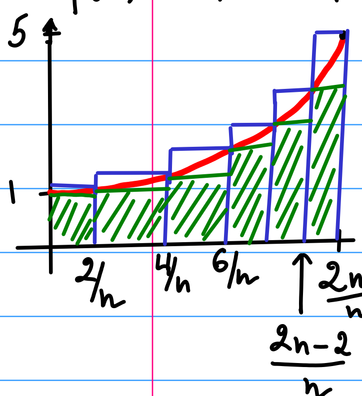
Ex. $1 + 2 + 3 + 4 + \dots + 1000 = \sum_{i=1}^{1000} i$

Ex. $1 + 8 + 27 + \dots + 1000 = \sum_{i=1}^{10} i^3$

$i = \quad 1 \quad 2 \quad 3 \quad \quad 10$

Areas. Sec. 5.1.

$f(x) = x^2 + 1$ on $[0, 2]$. Area under?



Estimate: divide $[0,2]$ into n subintervals

Each has length $\Delta x = \frac{2}{n}$

$$A_{upper} = \Delta x \cdot f\left(\tfrac{2}{n}\right) + \Delta x \cdot f\left(\tfrac{4}{n}\right) + \dots + \Delta x \cdot f\left(\tfrac{2n}{n}\right)$$

$$A_{Lower} = \Delta x \cdot f(0) + \Delta x \cdot f\left(\tfrac{2}{n}\right) + \dots + \Delta x \cdot f\left(\tfrac{2n-2}{n}\right)$$

$$A_{Lower} \le A_{real} \le A_{upper}$$

More rectangulars $\to$ better approximation.

as $n \to \infty$ $\quad \lim_{n\to\infty} A_{Lower} = \lim_{n\to\infty} A_{upper} = \lim_{n\to\infty} A_{real}$

Def. The area A that lies under $y = f(x)$ is

$$A = \lim_{n\to\infty} \left(f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \dots + f(x_n) \cdot \Delta x \right)$$

$x_1, \dots, x_n$ are called „sample points"

Definite Integral. Sec. 5.2.

Def. $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \quad , \text{ where}$$

$x_1, .., x_n$ – are sample points dividing $[a,b]$ into $n+1$ equal intervals of length $\Delta x = \dfrac{b-a}{n}$

– definite integral of f from a to b.

If limit exists $\to$ f is integrable on $[a,b]$

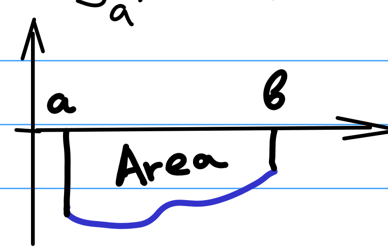
Notation:

$$\int_{\underset{\text{lower limit}}{a}}^{\overset{\text{upper limit}}{b}} \underbrace{f(x)}_{\text{integrand}}\,dx$$

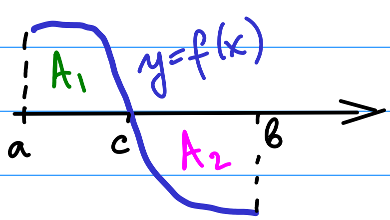
$$\sum_{i=1}^{n} f(x_i)\,\Delta x \longleftarrow \text{„Rimann sum"}$$

! $\int_a^b f(x)\,dx$ is a number (if exists).

! If $f(x)<0$ on $[a,b]$ then $\int_a^b f(x)\,dx = -\text{Area} < 0$



! if



$\Rightarrow$ $\int_a^c f(x)\,dx = A_1$

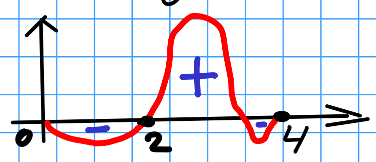
$\int_c^b f(x)\,dx = -A_2$

$\Rightarrow$ $\int_a^b f(x)\,dx \neq$ sum of areas But

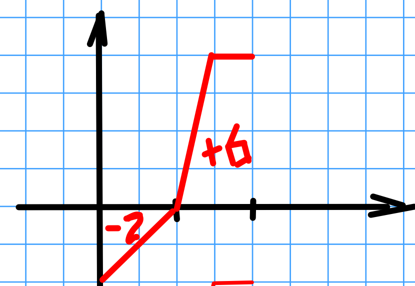
$= A_1 - A_2$

Lecture 33.

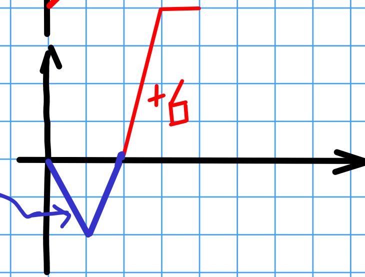
Ex. Draw the graph of a function s.t. $\int_0^4 f\,dx > 0$ and $\int_0^2 f\,dx < 0$.



Ex. $\int_0^4 f\,dx = 4$ $\qquad$ $\int_0^2 f\,dx = -2$



Ex. the same but **concavity is up** on $[0,2]$



Remark: Notice, that y'' is always zero here, but concavity is up.

THM. If f is **CONT.** or has only **FINITE #** of jump discont $\Rightarrow$ f is integrable.

* Then lim exists and any Rimann sum is an approximation of the integral $\Rightarrow$ take $\sum_i \Delta x\, f(\bar{x}_i)$ $\quad \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

! $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$ if $a < b$, Because in Rimann sum $\Delta < 0$.

! $$\int_a^a f(x)\,dx = 0$$

Computation of Integrals by Definition using Rimann Sums.

Properties of Σ. (App. E.)

THM. $\displaystyle\sum_{i=m}^{n}(a_i \pm b_i) = \sum_i a_i \pm \sum_i b_i$ $\qquad$ $\displaystyle\sum_{i=m}^{n} c a_i = c \cdot \sum_i a_i$

THM. $\displaystyle\sum_{i=1}^{n} 1 = n$ $\qquad$ $\displaystyle\sum_{i=1}^{n} c = n c$

$$\sum_{n=1}^{n} i = \frac{n\cdot(n+1)}{2} \qquad \sum_{n=1}^{n} i^2 = \frac{n\cdot(n+1)\cdot(2n+1)}{6}$$

$$\sum_{n=1}^{n} i^3 = \left(\frac{n\cdot(n+1)}{2}\right)^3$$

Computations. Sec. 5.2.

Ex. $\int_0^2 x^2+1\,dx =$

$[0, \frac{2}{n}]\,[\frac{2}{n}, \frac{4}{n}] \ldots [\frac{2n-1}{n}, \frac{2n}{n}]$ – n+1 sub intervals.

take $\int$ as lim of upper area $\Rightarrow$ as sampling points take $x_i^* = \frac{2i}{n}$

Rimann sum: $\sum_{i=1}^{n} \Delta x \cdot f(x_i^*) = \sum_{i=1}^{n} \frac{2}{n} \cdot f\left(\frac{2i}{n}\right) =$

$$= \sum_{i=1}^{n} \frac{2}{n} \cdot \left(\left(\frac{2i}{n}\right)^2 + 1 \right) = \sum_{i=1}^{n} \left(\frac{8i^2}{n^3} + \frac{2}{n} \right) = \frac{8}{n^3} \cdot \sum_{i=1}^{n} i^2 + \frac{2}{n} \cdot \sum_{i=1}^{n} 1 =$$

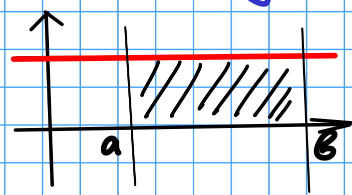
does not depend on i

$$= \frac{8}{n^3} \cdot \frac{n \cdot (n+1)(2n+1)}{6} + \frac{2}{n} \cdot n =$$

$$= \frac{4(n+1)(2n+1)}{3n^2} + 2 \xrightarrow[n\to\infty]{} \frac{8}{3} + 2 = \frac{14}{3}$$

Lecture 34.

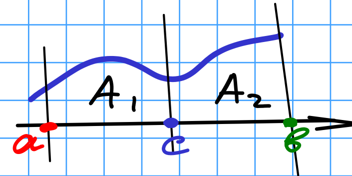
Laws of Integrals.

1. $\int_a^b c\,dx =$  $= c\cdot(b-a)$

2. $\int_a^b \left(f(x) \pm g(x)\right) dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

3. $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

4. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$



$A = A_1 + A_2$

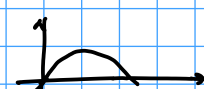
Comparison Properties.

1. $f(x) \geq 0$ on $[a,b]$ $\Rightarrow$ $\int_a^b f(x)\,dx \geq 0$

2. $f(x) \geq g(x)$ on $[a,b]$ $\Rightarrow$ $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

3. $m \leq f(x) \leq M$ on $[a,b]$ $\Rightarrow$ $m\cdot(b-a) \leq \int_a^b f(x)\,dx \leq M\cdot(b-a)$

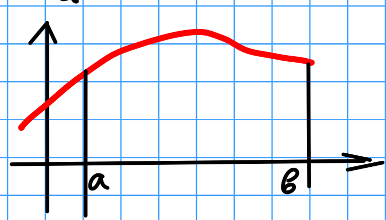
Ex. Estimate the value of $\int^{\pi} \sin x\,dx$



$0 \leq \sin x \leq 1$ on $[0,\pi]$ $\Rightarrow$ $0 \leq \int_0^{\pi} \sin x\,dx \leq \pi$

Fundamental Th. of Calculus. Sec. 5.3.

$\int_a^b f(t)\,dt$



let b is unknown and we want to know $\int_a^b$ for different b.

$\Rightarrow$ $g(x) = \int_a^x f(t)\,dt$

then $g(3) = \int_a^3 f(t)\,dt$

FTC 1 If $f(t)$ is cont.
$g(x) = \int_a^x f(t)\,dt$
$g(x)$ is cont. and differentiable

$\Bigg\}$ on $[a,b]$

$\Rightarrow$ $g'(x) = f(x)$

Ex. $\frac{d}{dx}\int_{-4}^{x} e^{2t}\cdot \cos^2(1-5t)\,dt = e^{2x}\cdot\cos^2(1-5x)$

Ex. $\frac{dg}{dx} = ?$ if $g(x) = \int_{-4}^{x^2} e^{2t}\,dt$

$g(x) = \text{integral}(x^2) \Rightarrow$ Usual Chain rule: let $u = x^2 \Rightarrow$

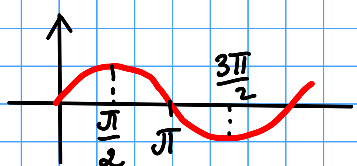
$\Rightarrow \frac{dg}{dx} = \frac{dg(u)}{du}\cdot\frac{du}{dx} = \frac{d}{du}\int_{-4}^{u} e^{2t}\,dt \cdot 2x \underset{\text{FTC}}{=} e^{2u}\cdot 2x =$

$= e^{2x^2}\cdot 2x$

FTC II. If f is CONT. on $[a,b] \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$

where F is any antider., i.e. $F' = f$.

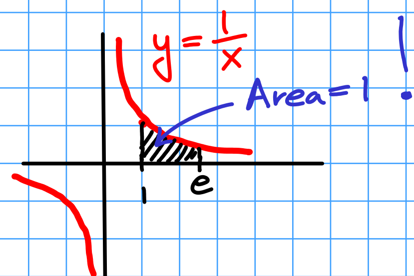
Ex. $\int_{\pi/2}^{\pi} \sin t\,dt = -\cos t\Big|_{\pi/2}^{\pi} = -(\cos\pi - \cos\frac{\pi}{2}) = -(-1) = 1$



Ex. $\int_{\pi/2}^{3\pi/2} \sin t\,dt = -\cos t\Big|_{\pi/2}^{3\pi/2} = -(\cos\frac{3\pi}{2} - \cos\frac{\pi}{2}) = 0.$

Ex. $\int_1^e \frac{1}{a}\,da = \ln|a|\Big|_1^e = \ln e - \ln 1 = 1$

$e \approx 2.71\,82\,81\,82$



Lecture 35.

Indefinite Integrals. Sec. 5.4.

Def. $\int f(x)\,dx = F(x)$ means $F'(x) = f(x)$ } just another notation for antiderivative

Ex. $\int x^4\,dx = \frac{x^5}{5} + C$

! $\int_a^b f(x)\,dx$ – number $\int f(x)\,dx$ – function

THM. $\int \underset{\nwarrow \text{const}}{a}\,dx = ax + C$, $\int kf(x) \pm g(x)\,dx = k\int f(x)\,dx + \int g(x)\,dx$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$ p. 398.

what if $n=1$? $\Rightarrow \int \frac{1}{x}\,dx = \ln|x| + C$

$\int e^x\,dx = e^x + C$ $\int a^x\,dx = \frac{a^x}{\ln a} + C$

$\int \sin x\,dx = -\cos x + C$ $\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$ $\int \csc^2 x\,dx = -\cot x + C$

$\int \sec x \cdot \tan x\,dx = \sec x + C$ $\int \csc x \cdot \cot x\,dx = -\csc x + C$

$\int \frac{dx}{1+x^2} = \arctan x + C$ $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

Ex. $\int \sqrt[4]{x}\,dx = \int x^{\frac{1}{4}}\,dx = \frac{x^{5/4}}{5/4}$

Ex. $\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2}$

Ex. $\int dy = \int 1\,dy = y + C$

Ex. $\int \sin\left(\frac{t}{2}\right) \cdot \cos\left(\frac{t}{2}\right) dt = \int \frac{\sin(t)}{2}\,dt = -\frac{\cos t}{2} + C$

Ex. Let $f''(x) = 2$ & $f(0)=0$ $f(1)=0$

$$f'(x) = \int 2\,dx = 2x + c_1$$

$$f(x) = \int 2x + c_1\,dx = x^2 + c_1x + c_2$$

$$f(0) = c_2 = 0 \qquad f(1) = 1 + c_1 + 0 = 0 \Rightarrow c_1 = -1$$

$$\Rightarrow f(x) = x^2 - x$$

<u>Read Net Change</u> p. 401.

Substitution Rule. Sec. 5.5.

Recall $\int (f(x))'\,dx = f(x) + C$ as $f(x)$ is obviously antiderivative to $f'(x)$

THM. If $g(x)$ is diff-able and $f(x)$ is cont. $\Rightarrow \int f(g(x)) \cdot \underbrace{g'(x)dx}_{dg(x)} = \int f(u)du$

Ex. $$\int \sec^2(\overset{u}{x^2}) \cdot \overset{du}{2x \cdot dx} = \int \sec^2(u)\,du = \tan u + C =$$

let $u = x^2 \Rightarrow du = 2x\,dx$

$$= \tan x^2 + C$$

Ex. $$\int_0^1 \sec^2(x^2) \cdot 2x\,dx = \tan x^2 \Big|_0^1 = \tan 1^2 - \tan 0^2 = \tan 1$$

Ex. $$\int \frac{2 \cdot x}{\sqrt{1-x^4}}\,dx = \int \frac{dx^2}{\sqrt{1-(x^2)^2}} \underset{u=x^2}{=\!=} \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + c$$

reminds us about $(\arcsin x)'$

$$= \arcsin x^2 + c$$

Ex. $$\int_0^1 \frac{x}{\sqrt{1-x^4}}\,dx = \frac{\arcsin x^2}{2}\Big|_0^1 = \frac{1}{2}(\arcsin 1 - \arcsin 0) =$$

notice!

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4}$$

Ex.

$$\int \frac{x^3\,dx}{\sqrt{1-x^4}} = \left[\begin{matrix}\text{notice that} \\ (x^4)' = 4x^3\end{matrix} \Rightarrow \begin{matrix} u = x^4 \\ du = 4x^3\,dx\end{matrix}\right] =$$

$$= \int \frac{\frac{1}{4}\,du}{\sqrt{1-u}} = \frac{1}{4}\int (1-u)^{-\frac{1}{2}}\,du = \frac{1}{4}\cdot\frac{(1-u)^{\frac{1}{2}}}{\frac{1}{2}} + C =$$

$$= \frac{1}{2}\cdot\sqrt{1-u} + C = \frac{\sqrt{1-x^4}}{2} + C$$

Ex.

$$\int_{-1}^{1} \frac{x^3\,dx}{\sqrt{1-x^4}} = \left.\frac{\sqrt{1-x^4}}{2}\right|_{-1}^{1} = 0 - 0 = 0.$$

Lecture 36.

Ex. $$\int e^{x^2}\cdot 3\cdot x\,dx = \left[\begin{array}{l} u = x^2 \\ du = 2x\,dx \\ \Rightarrow dx = \frac{du}{2x} \end{array}\right] = \int e^u \cdot \frac{3}{2}\,du = \frac{3e^u}{2} = 3e^{x^2}/2$$

Ex. $$\int_0^{\sqrt{\ln 2}} e^{x^2}\cdot 3x\,dx = \frac{3}{2}e^{x^2}\Big|_0^{\sqrt{\ln 2}} = \frac{3}{2}\left(e^{\ln 2} - e^0\right) = \frac{3}{2}(2-1) = \frac{3}{2}$$

Ex. $$\underline{\int \tan x\,dx} = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{d\cos x}{\cos x} = -\ln|\cos x| + c = \ln\frac{1}{|\cos x|} + c = \underline{\ln|\sec x| + c}$$ $\rightarrow$ memorize

Lecture 37.

Ex. $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx = ?$

Method 1: $\int \frac{x}{\sqrt{1+2x}}\,dx = \left[\begin{matrix} u = 1+2x \\ du = 2\,dx \end{matrix}\right] = \int \frac{(u-1)/2 \;\; \frac{du}{2}}{\sqrt{u}} =$

$= \int u^{-\frac{1}{2}} \cdot \frac{(u-1)}{4}\,du = \frac{1}{4}\int u^{1/2} - u^{-\frac{1}{2}}\,du = \frac{1}{4}\left(\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right) + C =$

$= \frac{1}{2}\left(\frac{(1+2x)^{3/2}}{3} - (1+2x)^{1/2}\right)$

$\Rightarrow \int_0^4 = \frac{1}{2}\left(\dots \right)\Big|_{x=4} - \frac{1}{2}\left(\dots \right)\Big|_{x=0} =$ too long!

Method 2 $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx = \frac{1}{4}\int_{u(0)}^{u(4)} u^{1/2} - u^{-\frac{1}{2}}\,du = \frac{1}{4}\int_1^9 \dots\,du =$

$= \frac{1}{2}\left(\frac{u^{3/2}}{3} - u^{1/2}\right)\Big|_1^9 = \frac{1}{2}\left(\frac{3^3}{3} - 3\right) - \frac{1}{2}\left(\frac{1}{3} - 1\right) =$

$= \frac{1}{2}\left(6 + \frac{2}{3}\right) = 3 + \frac{1}{3} = \frac{10}{3}$

$\Rightarrow$ **THM.** if g' is cont. on $[a,b]$ and f is cont. on the range of $u=g(x)$

$\Rightarrow \int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$

Ex. $\int_0^1 \frac{dt}{\cos^2 t\,\sqrt{1+\tan t}} = \left[\begin{matrix} u = \sqrt{1+\tan t} \\ du = \frac{1}{2\sqrt{1+\tan t}} \cdot \sec^2 t\,dt \end{matrix}\right] =$ ($\frac{1}{2\sqrt{1+\tan t}} = \frac{1}{2u}$)

$= \int_1^{\sqrt{2}} \frac{2u\,du}{\sec^2 t \cdot \cos^2 t\;\; u} = \int_1^{\sqrt{2}} 2\,du = 2(\sqrt{2}-1)$

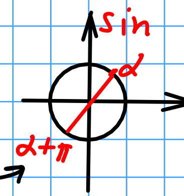
Ex. $\int_0^a x\sqrt{a^2-x^2}\,dx = \left[\begin{matrix} u = \sqrt{a^2-x^2} \\ du = \frac{-2x}{2u}\,dx \end{matrix}\right] = \int_{|a|}^0 x \cdot u \cdot \frac{-u}{x}\,du$

$= \int_0^{|a|} u^2\,du = \frac{u^3}{3}\Big|_0^{|a|} = \frac{|a|^3}{3}$

Ex. $$\int_{4/5}^{6/11} \frac{\cos(\pi/x)}{x^2}\,dx = \left[\begin{aligned} u &= \frac{\pi}{x} \\ du &= -\frac{\pi}{x^2}\,dx \end{aligned}\right] = \int_{\frac{5\pi}{4}}^{11\pi/6} \frac{\cos(u)\cdot x^2\cdot du}{(-\pi)\cdot x^2} =$$

$$= -\frac{1}{\pi}\int_{\frac{5\pi}{4}}^{\frac{11\pi}{6}} \cos u\,du = -\frac{1}{\pi}\left(\sin\left(\frac{11\pi}{6}\right) - \sin\left(\frac{5\pi}{4}\right)\right) =$$

$$= -\frac{1}{\pi}\left(\sin\left(2\pi - \frac{\pi}{6}\right) - \sin\left(\pi + \frac{\pi}{4}\right)\right) =$$



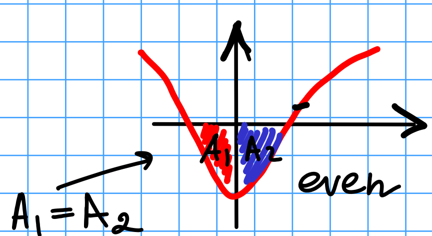
$$= -\frac{1}{\pi}\left(\sin 2\pi\cdot\cos\left(\frac{\pi}{6}\right) - \cos 2\pi\cdot\sin\frac{\pi}{6} - \left(-\sin\frac{\pi}{4}\right)\right) =$$

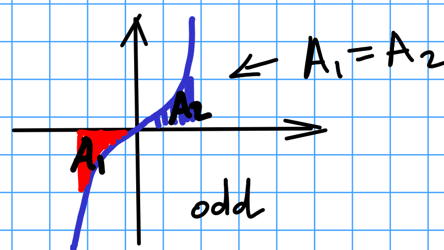
$$= -\frac{1}{\pi}\left(0 - \frac{1}{2} + \frac{\sqrt{2}}{2}\right) = \frac{1-\sqrt{2}}{2\pi}$$

THM. Let f be cont. on $[-a, a]$

(1) if f is even $\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(2) if f is odd $\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$.





Lecture 38.

Advanced problem: explain:

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$

$$\int \frac{dx}{\sqrt{1-(-x)^2}} = \left[\begin{array}{l} u = -x \\ du = -dx \end{array}\right] = \int - \frac{du}{\sqrt{1-u^2}} = \arccos(-x) + C$$

<u>Solution:</u> $\arccos(-x) = \pi - \arccos x + C$, and indeed:

$\arcsin x + \arccos x = \pi/2$, because

$\cos(\arcsin x + \arccos x) = 0.$, because

$\cos(\underbrace{\arcsin x}_{y}) = \sin(\underbrace{\arccos x}_{a})$ $\Rightarrow \cos a = x$

$\sin y = x$ $\qquad\qquad$ $\Downarrow$

$\cos y = \sqrt{1-x^2}$ $\qquad\qquad$ $\sin a = \sqrt{1-x^2}$

as $y \in [-\pi/2, \pi/2]$ $\qquad\qquad$ positive as $a \in [0, \pi]$.

Ex. If f is cont. and $\int_0^9 f(x)\,dx = 4$ find $\int_0^3 x f(x^2)\,dx$

$$\int_0^3 x f(x^2)\,dx = \left[\begin{array}{l} u = x^2 \\ du = 2x\,dx \end{array}\right] = \int_0^9 \frac{f(u)\,du}{2} = \frac{4}{2} = 2$$

Advanced problem: If f is cont, prove $\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)$

$$\int_0^{\pi/2} f(\cos x)\,dx = \left[\begin{array}{l} u = \cos x \\ du = -\sin x\,dx \end{array}\right] = \int_1^0 f(u)\,\frac{-du}{\sqrt{1-u^2}}$$

$\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-u^2}$

$\uparrow$ since $\sin x \geq 0$ for $x \in [0, \pi/2]$

$$\int_0^{\pi/2} f(\sin x)\,dx = \left[\begin{array}{l} u = \sin x \\ du = \cos x\,dx \end{array}\right] = \int_0^1 f(u) \cdot \frac{du}{\sqrt{1-u^2}}$$

//

Advanced problem: $\int_0^{\pi/2} \sin^2 x\,dx = ?$

$$\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx \Rightarrow 2\int_0^{\pi/2} \sin^2(x)\,dx = 1\cdot\left(\frac{\pi}{2} - 0\right)$$

$\Rightarrow$ Answer: $\frac{\pi}{4}$.

Exs: Even / Odd functions in Integrals.

Ex.

$$\int_{-3/\pi}^{3/\pi} \frac{\sec^2\left(\frac{1}{x}\right)}{x^2}\,dx = \left[\begin{matrix} f(x)=f(-x) \\ \text{even function}\end{matrix}\right] = 2\int_0^{3/\pi} \frac{\sec^2\left(\frac{1}{x}\right)}{x^2}\,dx$$

$$= \left[\begin{matrix} u=\frac{1}{x} \\ du=-\frac{1}{x^2}dx\end{matrix}\right] = 2\int_0^{\pi/3} -\sec^2 u\,du = 2\left(-\tan\left(\tfrac{\pi}{3}\right)+\tan(0)\right) = -2\sqrt{3}$$

Ex.

$$\int_{-\frac{1000\sqrt{2}}{\ln(2)}}^{\frac{1000\sqrt{2}}{\ln(2)}} \sin x\,dx = 0$$

Ex.

$$\int_{-1000000}^{1000000} \sin\frac{1}{x}\,dx = 0.$$

Ex.

$$\int_{-\ln(\sqrt{3})}^{\ln(\sqrt{3})} \underbrace{\arctan(x)}_{\text{odd}}\cdot\underbrace{x^2}_{\text{even}}\cdot\underbrace{\sec(x)}_{\text{even}}\,dx = 0.$$

One famous $\int$:

Ex.

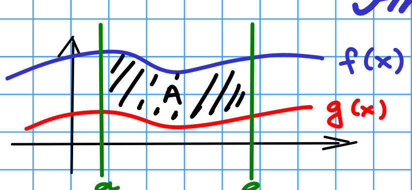
$$\int \sec x\,dx = \int \sec x\cdot\frac{\sec x+\tan x}{\sec x+\tan x}\,dx =$$

$$\left[\begin{matrix} u = \sec x + \tan x \\ du = \tan x\cdot\sec x + \sec^2 x\,dx = \sec x(\tan x + \sec x)\,dx = \\ = \sec x\cdot u\,dx\end{matrix}\right]$$

$$= \int \cancel{\sec x}\cdot\frac{\cancel{u}}{u}\cdot\frac{du}{\cancel{\sec x\cdot u}} = \ln|u| + C = \ln|\tan x + \sec x| + C$$

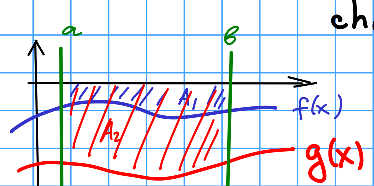
memorize

Area Between Curves. Sec. 6.1.



$$A = \int_a^b f(x) - g(x)\,dx$$

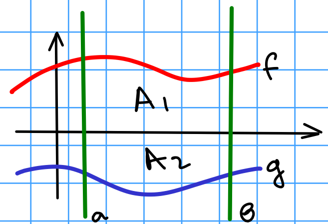
check



$$\int_a^b f(x) - g(x)\,dx = \underbrace{\int_a^b f\,dx}_{<0} - \underbrace{\int_a^b g\,dx}_{<0} =$$

$$= -A_1 - (-A_2) = A_2 - A_1$$

But // larger

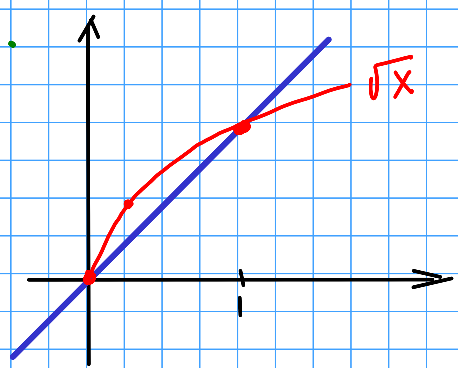


check: $\int_a^b f-g\,dx = \int_a^b f\,dx - \int_a^b g\,dx = A_1 - (-A_2)$
$= A_1 + A_2$

THM. The area A bounded between $y=f(x)$ and $y=g(x)$, $f(x) \geq g(x)$ on $[a,b]$ and lines $x=a$ and $x=b$ is

$$A = \int_a^b f-g\,dx.$$

Ex.



Area enclosed by: $y=\sqrt{x}$ and $y=x$

$a=0$, $b=1$

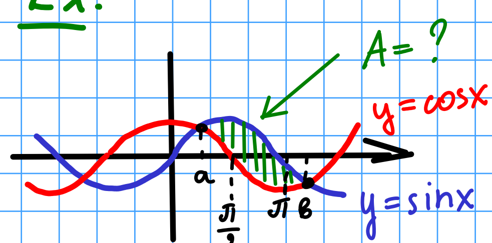
$\Rightarrow$ Area $= \int_0^1 \sqrt{x} - x\,dx =$

$$= \left(\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right)\Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

Lecture 39.

Informally: $A = \int_a^b \begin{pmatrix}\text{upper}\\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{lower}\\ \text{function}\end{pmatrix} dx$

Ex.



1) Intersection points: $\sin x = \cos x$, $k \in \mathbb{Z}$
$\tan x = 1 \Rightarrow$
$x = \frac{\pi}{4} + \pi k$

First two: $\frac{\pi}{4}$ and $\frac{5\pi}{4}$

2) $A = \int_{\pi/4}^{5\pi/4} \sin x - \cos x\,dx =$

$$= (-\cos x - \sin x)\Big|_{\pi/4}^{5\pi/4} = -\left(\cos\frac{5\pi}{4} + \sin\frac{5\pi}{4} - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)\right) =$$

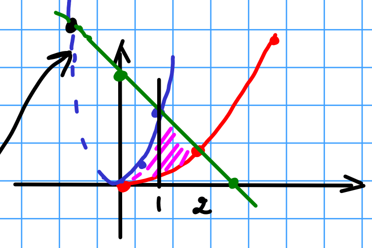
$$= -\left(\cos\left(\pi + \frac{\pi}{4}\right) + \sin\left(\pi + \frac{\pi}{4}\right) - \sqrt{2}\right) =$$

$$= -\left(-\cos\frac{\pi}{4} - 1\cdot\sin\frac{\pi}{4} - \sqrt{2}\right) = -\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - \sqrt{2}\right) = 2\sqrt{2}$$

Ex. (Splitting the area into two areas)

Find area enclosed by $y=\frac{1}{4}x^2$, $y=2x^2$, $x+y=3$ $\quad x \geq 0$

$y = 3-x$



1) Points of intersection:

$$\frac{1}{4}x^2 = 2x^2 \qquad 3-x = \frac{1}{4}x^2 \qquad 3-x = 2x^2$$

$$x^2 = 8x^2 \qquad 12-4x = x^2 \qquad 2x^2+x-3=0$$

$$7x^2 = 0 \qquad x^2+4x-12=0 \qquad D = b^2-4ac =$$

$$x=0 \qquad (x+6)\cdot(x-2)=0 \qquad = 1+4\cdot 2\cdot 3 = 25$$

$$(0,0). \qquad x=-6,\ x=2 \qquad x = \frac{-b\pm\sqrt{D}}{2a}$$

$$x_1 = \frac{-1+5}{4} = 1$$

we dont need this point

$$x_2 = \frac{-1-5}{4} = -\frac{3}{2}$$

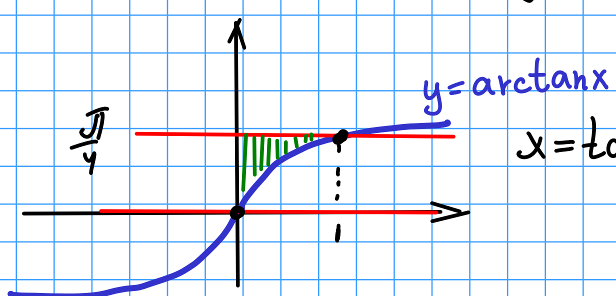
$$A = \int_0^1 2x^2 - \frac{x^2}{4}\,dx + \int_1^2 3-x-\frac{1}{4}x^2\,dx =$$

$$= 2\frac{x^3}{3}\Big|_0^1 - \int_0^2 \frac{x^2}{4}\,dx + 6 - \frac{x^2}{2}\Big|_1^2 =$$

$$= \frac{2}{3} - \frac{\cancel{2^3}^{\,2}}{\cancel{12}_{\,3}} + 6 - \left(2-\frac{1}{2}\right) = 6 - \frac{3}{2} = \frac{9}{2}.$$

Ex. "When x is a function of y."

Find area enclosed by $y=\arctan x$ and $y=\frac{\pi}{4}$.



$x = \tan y$

$$A = \int_0^1 \left(\frac{\pi}{4} - \arctan x\right) dx =$$

$$= \frac{\pi}{4} - \int_0^1 \arctan x\,dx = ?$$

try another way

$$A = \int_0^{\pi/4} \tan y\,dy = \ln|\sec y|\Big|_0^{\pi/4} = \ln\left|\frac{1}{\cos\frac{\pi}{4}}\right| - \ln\left|\frac{1}{\cos 0}\right| =$$

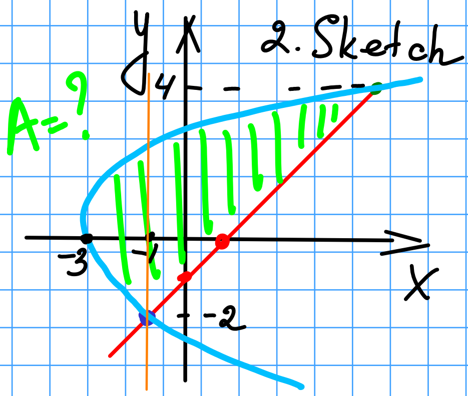
$$= \ln\left|\frac{2}{\sqrt{2}}\right| - \ln 1 = \ln\sqrt{2} = \frac{\ln 2}{2}$$

Lecture 40.

Ex. Determine the area of the region enclosed by $x=\frac{1}{2}y^2-3$ and $y=x-1$.

1) Intersection points: $x = y+1$

$$\frac{1}{2}y^2-3=y+1$$



$$y^2-2y-8=0$$

$$(y-4)\cdot(y+2)=0$$

$$y=-2 \Downarrow x=-1, \quad y=4 \Downarrow x=5$$

Our usual method: split into two and consider also $\frac{1}{2}y^2-3=x$ as two functions:

$$y=\sqrt{2(x+3)} \quad \text{and} \quad y=-\sqrt{2(x+3)}$$

$\Rightarrow$ sum of two ugly integrals ....

Instead: Consider w.r.t. „y":

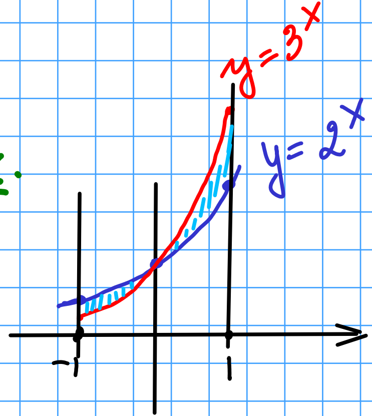
$$A=\int_{-2}^{4}(y+1)-\left(\frac{1}{2}y^2+3\right)dy=\left(\frac{y^3}{6}+\frac{y^2}{2}-2y\right)\Bigg|_{y=-2}^{y=4}=$$

$$=\frac{4^3}{6}+\frac{4^2}{2}-8-\left(\frac{-8}{6}+2\angle 4\right)=18$$

Informally, $\boxed{\int_c^d \begin{pmatrix}\text{right}\\ \text{function}\end{pmatrix}-\begin{pmatrix}\text{left}\\ \text{function}\end{pmatrix}dy}$

Lecture 41.

Ex.



Region enclosed by $y=2^x$, $y=3^x$ and $x=-1$, $x=1$

$$\int_{-1}^{0} 2^x - 3^x \, dx + \int_0^1 3^x - 2^x \, dx =$$

$$\text{BTW} = \int_{-1}^{1} |2^x - 3^x| \, dx \qquad = \left(\frac{2^x}{\ln 2} - \frac{3^x}{\ln 3}\right)\Bigg|_{-1}^{0} + \left(\frac{3^x}{\ln 3} - \frac{2^x}{\ln 2}\right)\Bigg|_{0}^{1} =$$

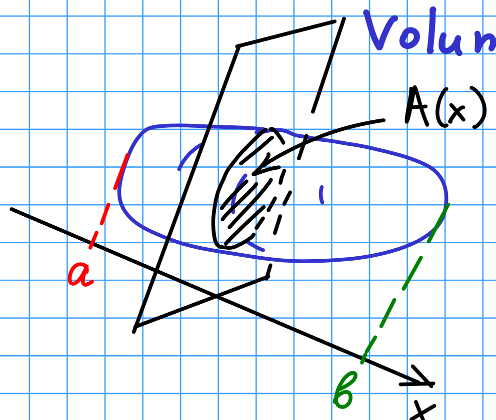
$$= \left(\cancel{\frac{1}{\ln 2}} - \frac{1}{\ln 3}\right) - \left(\frac{1}{2\ln 2} - \frac{1}{3\ln 3}\right) + \left(\frac{3}{\ln 3} - \cancel{\frac{2}{\ln 2}}\right) - \left(\frac{1}{\ln 3} - \cancel{\frac{1}{\ln 2}}\right) =$$

$$= \frac{3}{3\ln 3} - \frac{1}{2\ln 2} + \frac{1}{3\ln 3} = \frac{4}{3\ln 3} - \frac{1}{2\ln 2}$$

THM.

$$A = \int_a^b |f(x) - g(x)| \, dx$$

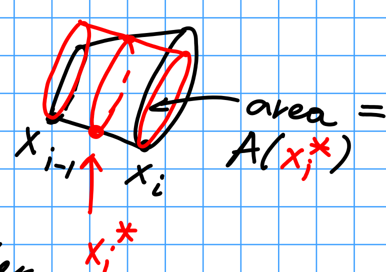
Volumes. Sec. 6.2.



„Slice" it into n pieces:

$$a < x_1 < x_2 < \dots < x_{n-1} < b$$

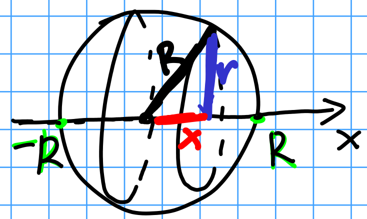
A piece then looks like this:



Take a „sampling point" and construct a cylinder using it $\Rightarrow$ volume $= \underbrace{\Delta x}_{\text{height}} \cdot A(x_i^*)$

$\Rightarrow$ Our Intuitive Idea of volume $\approx \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x \underset{\text{By definition of definite integral}}{=\!=} \int_a^b A(x)\,dx$

Ex. Sphere of radius R



Step 1: Area of the cross-section:

$$A(x) = \pi \cdot (r(x))^2 = \sqrt{R^2 - x^2}$$

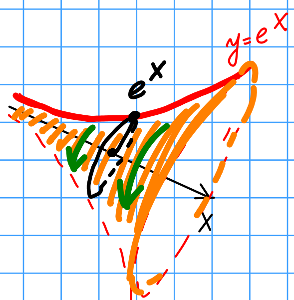
Step 2: $$V = \int_{-R}^{R} \pi \left(\sqrt{R^2-x^2}\right)^2 dx =$$

$$= \int_{-R}^{R} \pi \, (\underbrace{R^2 - x^2}_{\geq 0}) \, dx = \int_{-R}^{R} \pi R^2 - \pi \int_{-R}^{R} x^2 dx =$$

$$= \pi R^2 (R - (-R)) - \pi \left(\frac{R^3}{3} - \frac{(-R)^3}{3} \right) =$$

$$= 2\pi R^3 - \pi \cdot \frac{2R^3}{3} = \frac{4\pi R^3}{3}$$

Ex.



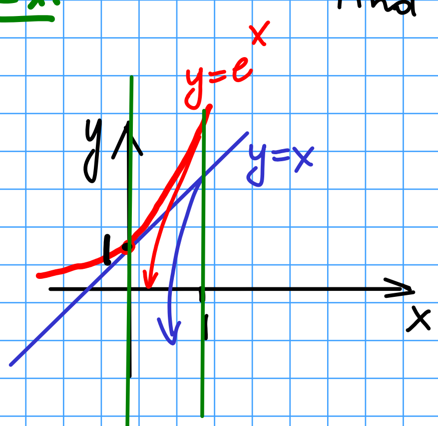
A machine part is made in a factory by rotation of $y = e^x$ around Ox from $x=0$ to $x=1$.
How much material (volume) would we need?

Step 1: Find cross-section function? $A(x) = \pi r^2 = \pi e^{2x}$

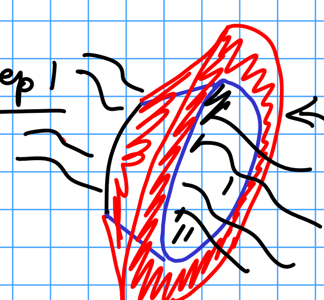
Step 2: $$V = \int_0^1 \pi e^{2x} dx = \left[\begin{matrix} u = 2x \\ du = 2dx \end{matrix} \right] = \frac{\pi}{2} e^u \Big|_0^2 = \frac{\pi}{2} e^2 - \frac{\pi}{2}$$

Lecture 42.

Ex. Find the volume of the solid obtained by rotation of the area enclosed by $y = e^x$ and $y = x+1$ and $x = 0$ and $x = 1$

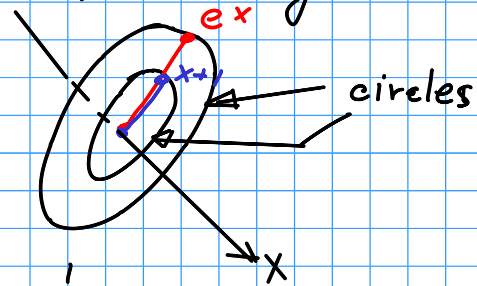


Step 1



We need to find the volume of red thing

Step 2: Area of the cross-section: at every x we have



$\Rightarrow A(x) = \pi \begin{pmatrix}\text{outer}\\ \text{radius}\end{pmatrix}^2 - \pi \begin{pmatrix}\text{inner}\\ \text{radius}\end{pmatrix}^2 = \pi (e^x)^2 - \pi \cdot (x+1)^2$

Step 3: $V = \int_0^1 A(x)\,dx = \pi \int_0^1 e^{2x} - (x+1)^2\,dx =$

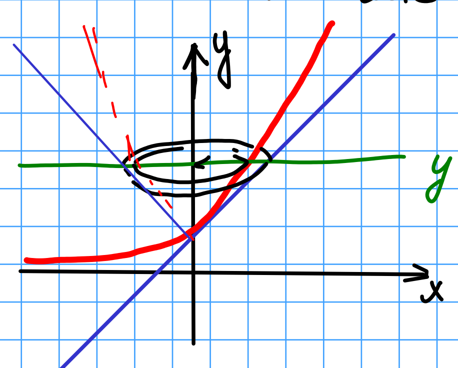
$$= \pi \int_0^1 e^{2x}\,dx - \frac{\pi (x+1)^3}{3}\Big/_0^1 = \frac{\pi}{2}e^2 - \frac{\pi}{2} - \frac{\pi \cdot 8}{3} = \pi\left(\frac{e^2}{2} - \frac{17}{6}\right)$$

Notation: If we rotate (= revolve) an area around the line $\Rightarrow$ we get **solids of revolution**

Remark: If cross-section is a disk $\Rightarrow A(x) = \pi (r(x))^2$

If — " — a washer $\Rightarrow A(x) = \pi (\text{outer } r)^2 - \pi (\text{inner } r)^2$

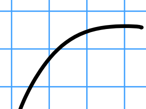
Ex. What if now we take $y = e^x$, $y = x+1$ and rotate around y-axis the area bounded by them and $y = \ln 3$



Step 1: $3 > e \approx 2.71828$

$\ln 3 > \ln e = 1$

as $\ln$ is $\uparrow$ function



Step 2

how functions look like w.r.t. y?

$x = \ln y$

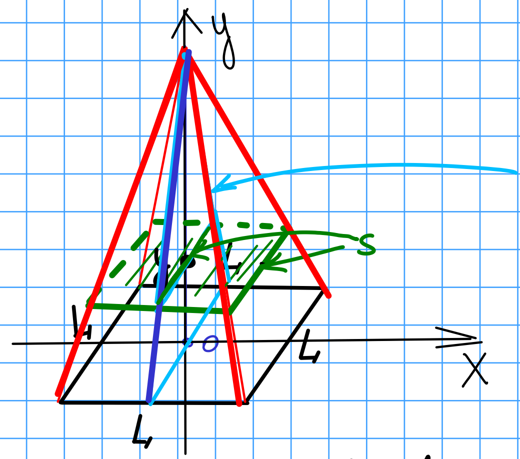
$x = y - 1$

$\Rightarrow$ inner radius $= \ln y$.

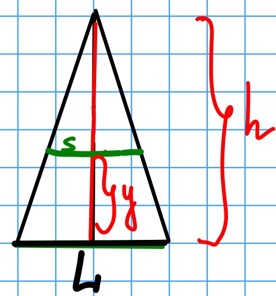
outer radius $= y$

$$\Rightarrow V = \pi \int_1^{\ln 3} (\ln y)^2\,dy - \pi \int_1^{\ln 3} (y-1)^2\,dy$$

Ex. Find the volume of a pyrimid whose base is a square with side L and whose height is h.



Place it so that the bottom sits on $y=0$.



$$\Rightarrow \frac{s}{L} = \frac{h-y}{h}$$

$$\Rightarrow s = \frac{L}{h}\cdot(h-y)$$

$$\Rightarrow A(y) = s^2 = \frac{L^2}{h^2}\cdot(h-y)^2$$

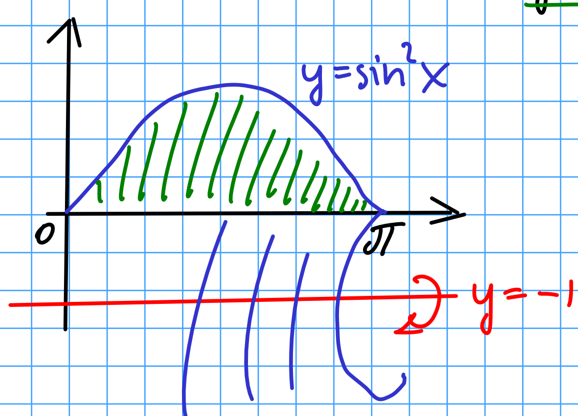
$$V = \int_0^h \frac{L^2}{h^2}(h-y)^2\,dy = \frac{L^2}{h^2}\int_0^h h^2 - 2hy + y^2\,dy =$$

$$= \frac{L^2}{h^2}\cdot\left(h^2\cdot h - hy^2\Big|_0^h + \frac{y^3}{3}\Big|_0^h\right) =$$

$$= \frac{L^2}{h^2}\left(h^3 - h^3 + \frac{h^3}{3}\right) = \frac{L^2 h}{3}$$

Lecture 43. Rotations about $y=a$ and $x=a$.

Ex. Rotate area $y=\sin^2 x$, $y=0$, $0 \le x \le \pi$ about $y=-1$.



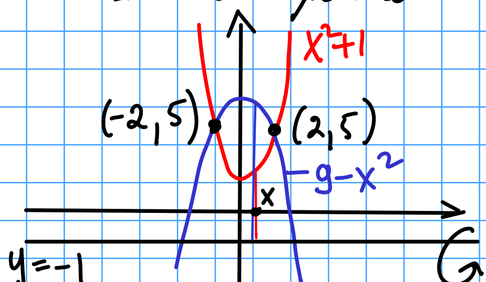
Outer radius: $\sin^2 x + 1$
inner radius: 1

$$V = \pi\int_0^\pi (\sin^2 x+1)^2 - 1\,dx =$$

$$= \pi\int_0^\pi \sin^4 x + 2\sin^2 x\,dx = \text{DO NOT compute.}$$

Ex. Region: $y=x^2+1$, $y=9-x^2$. About $y=-1$.

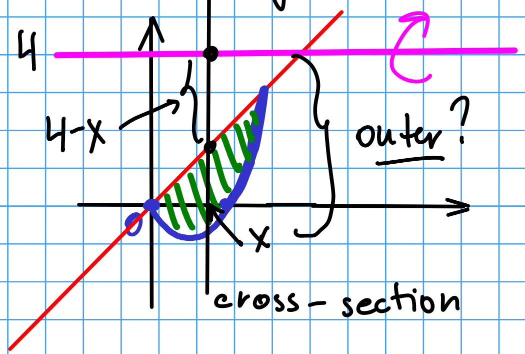
Intersec. points here easy: $x^2+1 = 9-x^2$
$2x^2 = 8$
$x = \pm 2 \Rightarrow y = 5$



$\Rightarrow$ Washer: inner radius: x^2+1+1
outer radius: $9-x^2+1$

$$V = \pi\int_{-2}^{2} (10-x^2)^2 - (x^2+2)^2\,dx = 256\pi$$

Ex. $y = x^2 - 2x$, $y = x$, rotate about $y = 4$



Intersec. Points: $x^2 - 2x = x$
$x^2 - 3x = 0$
$x(x-3) = 0$
$x = 0$ or $x = 3$

Inner Radius: $4 - x$
Outer Radius:

If $x^2 - 2x \le 0$, i.e. $x \in [0, 2]$
then outer radius $= 4 + |x^2 - 2x| =$
$= 4 - x^2 + 2x$

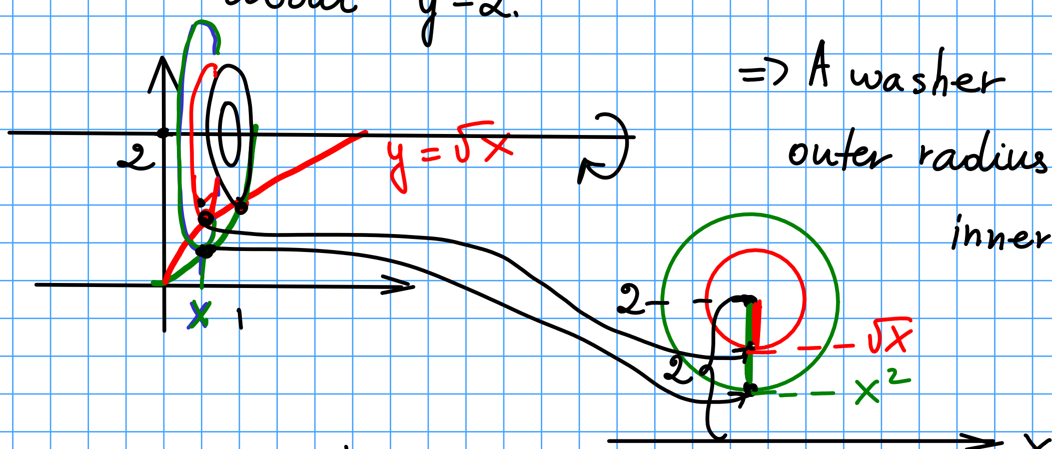
If $x^2 - 2x \geq 0$, i.e. $x \in [2, 3]$
then outer radius $= 4 - (x^2 - 2x)$

$\Rightarrow$ In both cases the same!

$A(x) = 4 - x^2 + 2x$

$$V = \pi \int_0^3 (4 - x^2 + 2x)^2 \, dx = \frac{153\pi}{5}$$

Ex. Find the volume of the solid obtained by rotating the area enclosed by $y = \sqrt{x}$, $y = x^2$ about $y = 2$.



$\Rightarrow$ A washer

outer radius: $2 - x^2$

inner radius: $2 - \sqrt{x}$

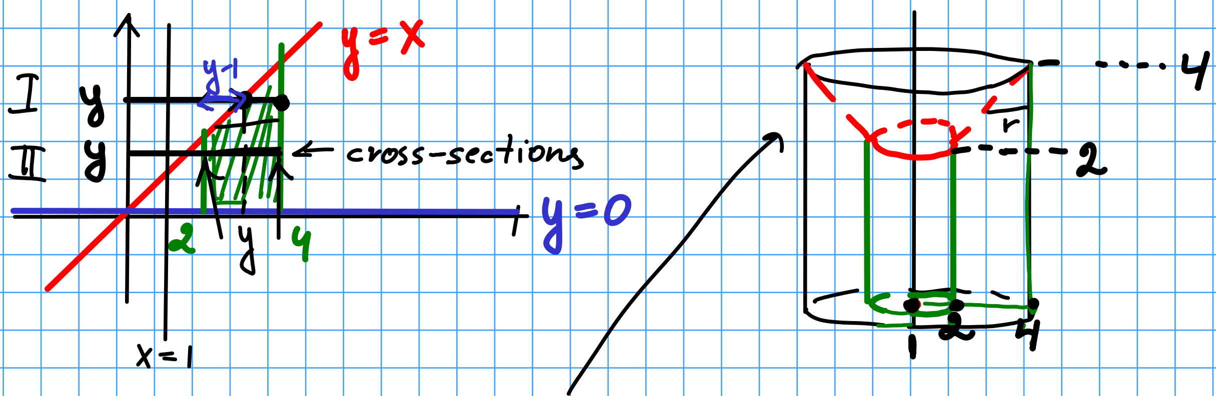
$$\Rightarrow V = \pi \int_0^1 (2 - x^2)^2 - (2 - \sqrt{x})^2 \, dx =$$

$$\pi \int_0^1 4 - 4x^2 + x^4 - 4 + 4\sqrt{x} + \underbrace{x}_{\text{as } x > 0} \, dx =$$

$$= \pi \left(\frac{x^5}{5} - \frac{4x^3}{3} + \frac{4x^{\frac{1}{2}+1}}{3/2} + \frac{x^2}{2} \right) \Bigg|_{x=0}^{x=1} = \frac{31}{30}$$

Splitting into 2 parts.

Ex. Volume of the region bounded by $y=x$, $y=0$, $x=2$, $x=4$ about $x=1$.



I: Cross section: outer radius $= 4-1=3$

inner radius $= y-1 = y-1$

$$\Rightarrow A_1(y) = \pi\cdot 3^2 - \pi\cdot(y-1)^2$$

$$V_1 = \int_2^4 9\pi - \pi(y-1)^2\,dy \overset{u=y-1}{=} 18\pi - \pi\int_1^3 u^2\,du = 18\pi - \pi\frac{u^3}{3}\Big|_1^3 =$$

$$= 18\pi - \pi\cdot\left(9-\frac{1}{3}\right) = 9\pi + \frac{1}{3}\pi$$

II: Cross section: outer radius $= 4-1=3$

inner radius $= 2-1=1$

$$A_2(y) = \pi\cdot 3^2 - \pi\cdot 1^2 = 8\pi$$

$$V_2 = \int_0^2 8\pi\,dy = 16\pi$$

$$\Rightarrow V = V_1 + V_2 = 25\frac{1}{3}\pi$$

Rotating around $y=a$.

Ex. $y = 2\sqrt{x-1}$, $y = x-1$ about $x=-1$

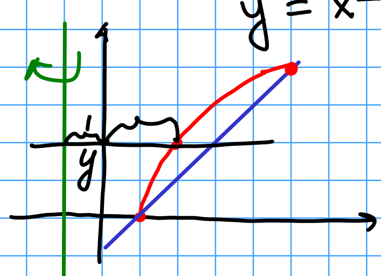
① Intersection points:

$2\sqrt{x-1} = x-1$

$2 = \sqrt{x-1}$, $x \neq 1$

$4 = x-1$

$x=5 \to y=4$



Cross-section of this washer: inner radius:

$$y = 2\sqrt{x-1}$$

$$\frac{y}{2} = \sqrt{x-1} \Rightarrow 1 + \frac{y^2}{4} = x$$

$\Rightarrow$ inner radius: $1 + \frac{y^2}{4} + 1$

outer radius:

$y = x - 1 \Rightarrow x = y + 1$ $\Rightarrow$ $y + 1 + 1$

(2) $$A(y) = \pi (y+2)^2 - \pi \cdot \left(2 + \frac{y^2}{4}\right)^2$$

(3) $$\int_0^4 A(y)\, dy = \frac{96\pi}{5}$$

Lecture 44. Review.

Limits.

Squeeze:

Ex. $\lim\limits_{x\to 0-} \sin x \cdot \cos\frac{1}{x} = 0\cdot\infty$ – indeterminate form

$$-1 \le \cos\frac{1}{x} \le 1$$

$$\underbrace{-\sin x}_{\downarrow\, 0} \le \cos\frac{1}{x} \le \underbrace{\sin x}_{\downarrow\, 0}$$

Answer: 0

At „∞"

Ex. $$\lim_{x\to-\infty}\frac{1+x^3}{-10x^3+7} = \lim_{x\to-\infty}\frac{x^3\left(\frac{1}{x^3}+1\right)}{x^3\left(-10+\frac{7}{x^3}\right)} = -\frac{1}{10}$$

L'H.

Ex. $$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty}\frac{1/x}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0.$$

Lecture 45.

Ex. $\lim\limits_{x\to\infty} \sin\left(\frac{1}{x}\right)\cdot x = [0\cdot\infty] = \lim\limits_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{1/x} = \left[\frac{0}{0}\right]$

$\frac{1}{x} \to 0$

$= \lim\limits_{x\to\infty} \frac{\left(\frac{1}{x}\right)'\cdot\cos\left(\frac{1}{x}\right)}{(1/x)'} = \lim\limits_{x\to\infty}\cos\left(\frac{1}{x}\right) = 1$

Ex. (For L'H it is not necessarily $x\to\infty$ or $x\to 0$)

$\lim\limits_{x\to 2-} \frac{\sin(\ln(x^2-3))}{x-2} = \left[\frac{0}{0}\right] = \lim\limits_{x\to 2-} \frac{\cos(\ln(x^2-3))\cdot 2x}{x^2-3} = 4$

Notice that squeeze thm is not working here:

$-1 \le \sin(\dots) \le 1$

$\frac{-1}{x-2} \geqslant \frac{\sin(\dots)}{x-2} \geqslant \frac{1}{x-2}$

$\frac{-1}{x-2} \to -\infty$, $\frac{1}{x-2} \to +\infty$

change signs because $x\to 2-$ an so $x-2<0$.

Ex. (Not L'H) $\lim\limits_{x\to 1} \arccos\left(\frac{1-\sqrt{x}}{x-1}\right) =$

$= \lim\limits_{x\to 1} \arccos\left(-\frac{\sqrt{x}-1}{x-1}\right) = \lim\limits_{x\to 1}\arccos\left(-\frac{1}{\sqrt{x}+1}\right) =$

$= \arccos\left(-\frac{1}{2}\right) = \pi - \arccos\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Ex. $\lim\limits_{x\to 1+} (2-x)^{\tan(\pi x/2)} = 1^{-\infty} \leftarrow$ undetermined form

$y = (2-x)^{\tan(\pi x/2)}$

$\lim\limits_{x\to 1+} \ln y = \lim\limits_{x\to 1+} \tan\left(\frac{\pi x}{2}\right)\cdot\ln(2-x) = [-\infty\cdot 0] =$

$= \lim\limits_{x\to 1+} \frac{\ln(2-x)}{\cot\left(\frac{\pi x}{2}\right)} \overset{L'H}{=} \lim\limits_{x\to 1+} \frac{\frac{1}{2-x}\cdot(-1)}{-\csc^2\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}} =$

$= \frac{2}{\pi}\cdot\lim\limits_{x\to 1+} \frac{\sin^2\left(\frac{\pi x}{2}\right)}{2-x} = \frac{2}{\pi}\cdot\frac{1}{1} = \frac{2}{\pi}$

$\ln(\text{Answer}) = \frac{2}{\pi} \Rightarrow$ Answer: $e^{2/\pi}$

Indeterminate Forms of Types 1^∞, 0^0, ∞^0.

Ex. $\lim\limits_{x\to 0+} \underbrace{(\tan 2x)^x}_{=y} = 0+^{0+}$ indeterminate form

$$\lim_{x\to 0+} \ln y = \lim_{x\to 0+} x \ln(\tan 2x) = [0\cdot(-\infty)] =$$

$$= \lim_{x\to 0+} \frac{\ln(\tan 2x)}{1/x} = \left[\frac{-\infty}{\infty}\right] [L'H] = \lim_{x\to 0+} \frac{2\sec^2(2x)}{\tan(2x)\cdot\left(-\frac{1}{x^2}\right)} =$$

$$= \lim_{x\to 0+} \frac{-2\cdot 1/\cos^2(2x)\cdot x^2}{\frac{\sin(2x)}{\cos(2x)}} = \lim_{x\to 0+} \frac{-2x^2}{\sin(2x)\cdot\cos(2x)} =$$

$$= \lim_{x\to 0+} \frac{-4x^2}{\sin(4x)} = \left[\frac{0}{0}\right] = \lim_{x\to 0+} \frac{-8x}{4\cos 4x} = 0$$

$\Rightarrow$ Answer: $e^0 = 1$

Ex. $\lim\limits_{x\to \frac{\pi}{2}-} (\tan x)^{\cos x} = (+\infty)^0$ – indeterm. form.

$$\lim_{x\to \frac{\pi}{2}-} \cos x\cdot \ln(\tan x) = \lim_{x\to \frac{\pi}{2}-} \frac{\ln(\tan x)}{\sec x} =$$

$$= \lim_{x\to \frac{\pi}{2}-} \frac{\sec^2 x}{\tan x\cdot\tan x\cdot\sec x} = \lim_{x\to \frac{\pi}{2}-} \frac{\frac{1}{\cos x}}{\frac{\sin^2 x}{\cos^2 x}} = \lim_{x\to \frac{\pi}{2}-} \frac{\cos x}{\sin^2 x} = 0$$

$\Rightarrow$ Answer: $e^0 = 1$

Computation of limits by definition.

Ex. Compute $\left.\frac{d(e^x)}{dx}\right|_{x=1}$ by definition

$$\lim_{h\to 0} \frac{e^{h+1}-e^1}{h} = e \lim_{h\to 0} \frac{e^h - 1}{h} = e\cdot 1 = e$$

Lecture 46.

Ex. Compute $\left.\dfrac{d\left((x+1)^2\right)}{dx}\right|_{x=1}$

$$\lim_{h\to 0}\frac{(h+1+1)^2-2^2}{h}=\lim_{h\to 0}\frac{h^2+4h}{h}=\lim_{h\to 0}(h+4)=4$$

Check: $\left.2(x+1)\right|_{x=1}=4.$

Ex. $\left.\dfrac{d(f(x))}{dx}\right|_{x=a}=\lim\limits_{h\to 0}\dfrac{2^{h+1}-2}{h}$

What is $f(x)$ and a ? $\left(2^x \text{ and } 1\right)$

Ex. $f'(a)=\lim\limits_{h\to 0}\dfrac{(h+2)^3-8}{h}$

$f(x)=?$ $[=x^3]$

$a=?$ $[=2]$

Ex. Does $\lim\limits_{h\to 0}\dfrac{(h+2)^3-8}{h}$ equals to $\lim\limits_{h\to 0}(12+4\cdot h)$

$$=\lim_{h\to 0}\frac{h^3+3h^2\cdot 2+3\cdot h\cdot 2^2+8-8}{h}=12$$

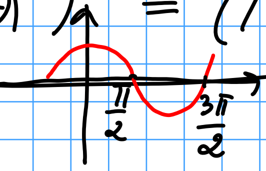
Answer: yes.

Review. Derivatives.

Ex. $$\left(\arctan(\ln(\sqrt{x}+1))\right)'=\frac{\frac{1}{\sqrt{x}+1}\cdot\frac{1}{2}\frac{1}{\sqrt{x}}}{1+(\ln(\sqrt{x}+1))^2}$$

Ex. $$\left(\sqrt{1-(2\cos\theta\,\sin\theta)^2}\right)'=(|\cos\theta|)'=(-\cos\theta)'=\sin\theta$$

$\frac{\pi}{2}\le\theta\le\frac{3\pi}{2}$



Ex. $$\left(\cos(\arccos x+16\pi)\right)'=\left(\cos(\arccos x)\right)'=(x)'=1$$

Ex. $$\left(\frac{\cos(x+2\pi)}{\sin(x+4\pi)}\cdot\frac{1}{\csc(-x)}\right)' = (-\cos x)' = \sin x$$

Ex. $$\left(\frac{x-1}{\sqrt{x}+1}\right)' = (\sqrt{x}-1)' = \frac{1}{2}\frac{1}{\sqrt{x}}$$

Ex. $$\left(\cos^{-1}\left(\cos\frac{\pi}{5}+x\right)\right)' = -\frac{1}{\sqrt{1-\left(\cos\frac{\pi}{5}+x\right)^2}}$$

Ex. $$\Bigg(\underbrace{\frac{\sqrt{x}\cdot(\sin(x)+1)^{1/4}}{\cos^{-1}(\sqrt{x})}}_{y}\Bigg)'$$

careful

$$(\ln(y))' = \left(\frac{1}{2}\ln x + \frac{1}{4}\cdot\ln(\sin x+1) - \ln(\arccos(\sqrt{x}))\right)' =$$

$$\frac{y'}{y} = \underbrace{\frac{1}{2x} + \frac{\cos x}{4(\sin x+1)} - \frac{-\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x}}}{\arccos(\sqrt{x})}}$$

$$\Rightarrow y' = \quad\cdot$$

Ex. $$\Big(\underbrace{x^{x^2}}_{y}\Big)'$$

$$\underbrace{(\ln y)'}_{\frac{y'}{y}} = (x^2\ln x)' = 2x\ln x + x^2\cdot\frac{1}{x}$$

$$\Rightarrow y' = (2x\ln x + x)\cdot x^{x^2}$$

Lecture 47.

Ex. $$\Big(\underbrace{\csc x^{\csc x}}_{y}\Big)$$

$$(\ln y)' = \frac{y'}{y} = (\csc x\cdot\ln(\csc x))' = -\csc x\cdot\cot x\cdot\ln(\csc x) +$$

$$+\frac{1}{\csc x}\cdot(-\csc x\cdot\cot x)$$

$$\Rightarrow y' = (1-\csc x\cdot\cot x\cdot\ln(\csc x))\cdot\csc x^{\csc x}$$

Min-Max Problems.

Ex. Critical numbers (= points) of $y=\ln(x^2-4)$?

$$y' = \frac{2x}{x^2-4}$$

1) $y'=0 \iff x=0$

2) y' DNE $\iff x=\pm 2$

But! Domain of y itself: $x^2-4>0$, $x^2>4$ $\Rightarrow$ no critical points

Ex. Critical numbers (= points) of $y=\frac{x}{x-1}$

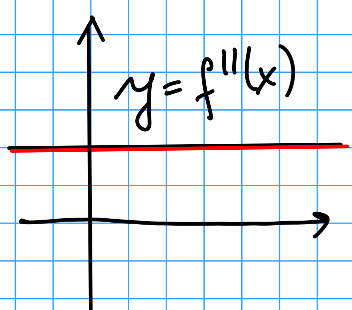
$$y' = \frac{1\cdot(x-1)-x\cdot 1}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

1) $y'=0$: never

2) y' DNE: $x=1 \notin$ Domain

$\Rightarrow$ no critical points.

Ex.



Suppose $x=0$ is the critical point of $y=f(x)$ and

Does $x=0$ local max, min or nothing?

Why? $f''(x)\big|_{x=0} > 0 \Rightarrow$ concave up $\Rightarrow$ ◡ at $x=0$ $\Rightarrow$ local minimum

Notice: if we know that $x=0$ is the only critical point and $f''(x)$ everywhere on $(-\infty,\infty) \Rightarrow$

$\Rightarrow x=0$ — global minimum.

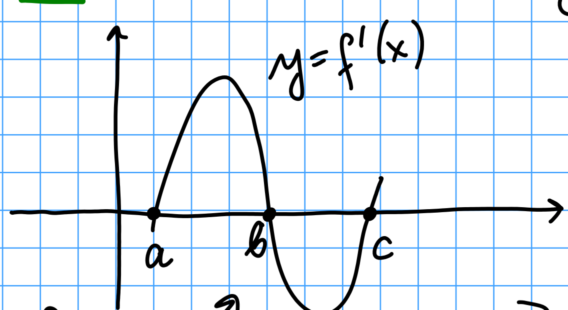
Notice: if on interval we should check end points also.

Ex. Based on the graph below does $x=b$

1) abs. min. on $[a,c]$

2) inflection point

3) abs. max on $[a,c]$



$\Rightarrow$

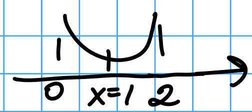
Ex. Find max of $y = \frac{x^2 - 2x}{3}$ on $(0, 2)$

$y' = \frac{1}{3}(2x-2) = \frac{2}{3}(x-1) = 0 \iff x = 1$

Method 1

Second derivative test: $y'' = \frac{2}{3} > 0 \Rightarrow$ concavity up

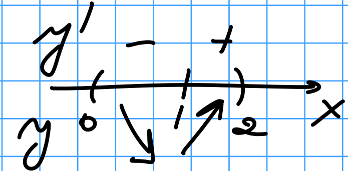
$\Rightarrow$ global minumum



Method 2 First derivative test:

for $x > 1$ $y' > 0 \Rightarrow$

$x < 1$ $y' < 0 \Rightarrow$



Sigma Notation.

Ex. $2 + 4 + 6 + \ldots + 1000 = \sum_{i=1}^{50} 2i$

Ex. $5 + 7 + 9 + \ldots + 121 = \sum_{i=2}^{60} 2i+1$

Integrals.

Ex.

$$\frac{d}{dt} \int_a^{(sc s t)^2} \tan(\sqrt{s})\, ds = \frac{d}{dt} F\big(\underbrace{(scst)^2}_{u}\big) =$$

$$= \frac{dF(u)}{du} \cdot \frac{du}{dt} \overset{FTC\ I}{=\!=} \tan(|scct|) \cdot (-\csc t \cdot \cot t) \cdot$$

$$\cdot 2\csc t$$

Ex.

$$\frac{d}{dt} \int_{\cos^{-1}(t)}^{\csc t} s^2\, ds = \frac{d}{dt}\left(\int_{\cos^{-1}(t)}^{0} s^2 ds + \int_0^{\csc t} s^2 ds \right) =$$

$$= \frac{d}{dt}\left(-\int_0^{\cos^{-1}(t)} s^2 ds + \int_0^{\csc t} s^2 ds \right) = -(\arccos(t))^2 \cdot \frac{-1}{\sqrt{1-t^2}} +$$

$$+ (\csc t)^2 \cdot (-\csc t \cdot \cot t)$$