MAT 1320 A Fall 2011 October 5th, 8:30 Prof. Nevins

Solution:

1. [2 points] Solve for x: $e^{3x+5} = 2^x$. (Hint: use ln.)

Solution:

$$\begin{aligned} e^{3x+5} = 2^x &\Rightarrow \ln(e^{3x+5}) = \ln(2^x) \Rightarrow 3x + 5 = x\ln(2) \\ &\Rightarrow 3x - x\ln(2) = -5 \Rightarrow x(3 - \ln(2)) = -5 \\ &\Rightarrow x = \frac{-5}{3 - \ln(2)} \end{aligned}$$

Grading scheme: Deduct 1 point for each error (to a minimum of 0).

2. [3 points] Evaluate the following derivatives:

(a) $\frac{d}{dx}\left(3x^4 + 7e^x - \sqrt{x} + \frac{5}{x^2}\right) =$

Solution:

We use the linearity of the derivative, that is: $\frac{\mathrm{d}}{\mathrm{d}x}(f(x) + g(x)) = \frac{\mathrm{d}f}{\mathrm{d}x}(x) + \frac{\mathrm{d}g}{\mathrm{d}x}(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}(cf(x)) = c\frac{\mathrm{d}f}{\mathrm{d}x}(x)$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(3x^4 + 7e^x - \sqrt{x} + \frac{5}{x^2}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(3x^4\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(7e^x\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1/2}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{5}{x^2}\right) \\ &= 3\frac{\mathrm{d}}{\mathrm{d}x}\left(x^4\right) + 7\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1/2}\right) + 5\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-2}\right) \\ &= 3\left(4x^3\right) + 7e^x - \left(\frac{1}{2}x^{-1/2}\right) + 5\left(-2x^{-3}\right) \\ &= 12x^3 + 7e^x - \frac{1}{2}x^{-1/2} - 10x^{-3} \end{aligned}$$

Grading scheme: Each derivative is worth 0.5; but deduct 1 point for "+c" (minimum 0)

(b) $\dfrac{d}{dx}\displaystyle\int_2^x \left(\frac{1}{\sqrt{t}} - e^t\right)\, dt =$

Solution:
We apply Part I of the Fundamental Theorem of Calculus, that is, $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_a^x f(t)\,\mathrm{d}t = f(x)$ whenever f is continuous. Thus

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_2^x \left(\frac{1}{\sqrt{t}} - e^t\right)\,\mathrm{d}t = \frac{1}{\sqrt{x}} - e^x$$

Grading scheme: 1 point for correct answer (independent of method use to obtain it); deduct 0.5 for "+c"; deduct 0.5 for answer in t rather than x (minimum 0)

3. [2 points] Evaluate $\displaystyle\int_1^4 \left(7e^x - \sqrt{x} + \frac{6}{x^4}\right) dx.$

Solution:
We first use the linearity of the integral, that is, $\displaystyle\int_a^b (f(x)+g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$ and $\displaystyle\int_a^b (cf(x))\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$. Then, we use Part II of the Fundamental Theorem to evaluate the resulting simpler integrals.

$$\begin{aligned}
\int_1^4 \left(7e^x - \sqrt{x} + \frac{6}{x^4}\right)\,\mathrm{d}x &= 7\int_1^4 e^x\,\mathrm{d}x - \int_1^4 x^{1/2}\,\mathrm{d}x + 6\int_1^4 x^{-4}\,\mathrm{d}x \\
&= 7(e^x)\Big|_1^4 - \left(\frac{2}{3}x^{3/2}\right)\Bigg|_1^4 + 6\left(-\frac{1}{3}x^{-3}\right)\Bigg|_1^4 \\
&= 7(e^4 - e) - \frac{2}{3}\left(4^{3/2} - 1\right) - 2\left(4^{-3} - 1\right) \\
&= 7(e^4 - e) - \frac{14}{3} + \frac{63}{32} = 7(e^4 - e) - \frac{259}{69}(\sim 359.405)
\end{aligned}$$

Grading scheme: 0.5 for each of the 3 correct anti-derivatives and 0.5 for correct final

answer. Either decimal approximation or exact answer accepted. Write "simplify" if student does not simplify; points will be deducted on Midterm #2.

4. [2 points] Which of the following expressions is the **right Riemann sum with** n **subintervals** for estimating the value of the integral $\int_0^1 \sin(\sqrt{x})\,dx$?

A: $\sum_{i=1}^{n} \sin\left(\sqrt{\frac{i}{n}}\right)\frac{1}{n}$ **B**: $\sum_{i=0}^{n-1} \sin\left(\sqrt{\frac{i}{n}}\right)\frac{1}{n}$ **C**: $\sum_{i=1}^{n} \sin\left(\sqrt{\frac{i-1}{n}}\right)\frac{1}{n}$

D: $\sum_{i=1}^{n} \sin\left(\sqrt{\frac{i+1}{n}}\right)\frac{1}{n}$ **E**: $\sum_{i=1}^{n} \sin\left(\sqrt{\frac{i}{n}}\right)\frac{1}{2n}$ **F**: $\sum_{i=0}^{n-1} \sin\left(\sqrt{\frac{i}{n}}\right)\frac{1}{2n}$

Solution:
Since

$$f(x) = \sin(\sqrt{x})$$
$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

and

$$x_i = \frac{i}{n} \quad \text{for} \quad i = 0, 1, 2, \ldots, n \ ,$$

we have

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \sin\left(\sqrt{\frac{i}{n}}\right)\frac{1}{n}$$

The correct answer is thus A.

Grading scheme: 2 points; no partial credit. Only one answer may be circled.

5. [2 points] Consider the velocity data of a car in the table below.

t (s)	0	2	4	6	8
v (m/s)	14	10	8	-2	-4

Compute the **left** Riemann sum with 4 subintervals of $\int_0^8 v(t)\ dt$. Include the unit of measurement in your answer.

Solution:
Since $\Delta x = 2$, we have

$$L_4 = (v(0) + v(2) + v(4) + v(6))\,\Delta x = 2\,(14 + 10 + 8 - 2) = 60 \text{ m}$$

Grading scheme: 1 for correct formula/expression, 0.5 for correct number, 0.5 for correct units.

6. [2 points] Suppose

$$\int_0^9 f(x)\ dx = 3 \qquad \text{and} \qquad \int_7^9 f(x)\ dx = -4.$$

Evaluate $\displaystyle\int_0^7 (3f(x) + 2)\ dx$.

Solution:
We apply the property that $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$.

Here, we have

$$\int_0^7 f(x)\,\mathrm{d}x = \int_0^9 f(x)\,\mathrm{d}x - \int_7^9 f(x)\,\mathrm{d}x = 3 - (-4) = 7$$

Thus, by linearity of the integral, we have

$$\int_0^7 (3f(x) + 2)\,\mathrm{d}x = 3\int_0^7 f(x)\,\mathrm{d}x + \int_0^7 2\,\mathrm{d}x = 3(7) + (2x)\Big|_0^7 = 21 + 14 = 35.$$

Grading scheme: 0.5 for formula to find the required integral of f; 0.5 for correct value; 0.5 for expanding the integral as a sum; 0.5 for final answer.

7. [4 points]

(a) State the definition of the derivative of a function f at an arbitrary point x in the domain.

Solution:
The derivative of f at a point x in the domain of f is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists.

Grading scheme: 1 point. Write "if limit exists" if this is missing but do not deduct.

(b) Use the definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{2}{3x-1}$. Write your solution correctly, logically, and clearly.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left(\dfrac{2}{3(x+h)-1} - \dfrac{2}{3x-1} \right)}{h} \\ &= \lim_{h \to 0} \frac{2(3x-1) - 2(3(x+h)-1)}{h(3(x+h)-1)(3x-1)} = \lim_{h \to 0} \frac{-6h}{h(3(x+h)-1)(3x-1)} \\ &= \lim_{h \to 0} \frac{-6}{(3(x+h)-1)(3x-1)} = \frac{-6}{(3x-1)^2} \end{aligned}$$

whenever $x \neq 1/3$ since $3(x+h) - 1 \to 3x - 1$ as $h \to 0$.

Grading scheme: 1 for correctly plugging the function into the formula for the difference quotient, 1 point for successful simplification of the difference quotient, 1 for correct final limit.

8. [3 points] We are given the function

$$g(x) = \begin{cases} x^2 & \text{if } 0 \leq x < 2 \\ 4 & \text{if } 2 \leq x < 4 \\ 12 - 2x & \text{if } 4 \leq x \leq 8. \end{cases}$$

Evaluate $\int_0^8 g(x)\, dx$.

Solution:

We split this integral into regions upon which we can find an anti-derivative, and then apply the Evaluation Theorem.

$$
\begin{aligned}
\int_0^8 g(x)\,\mathrm{d}x &= \int_0^2 g(x)\,\mathrm{d}x + \int_2^4 g(x)\,\mathrm{d}x + \int_4^8 g(x)\,\mathrm{d}x \\
&= \int_0^2 x^2\,\mathrm{d}x + \int_2^4 4\,\mathrm{d}x + \int_4^8 (12-2x)\,\mathrm{d}x \\
&= \left(\frac{x^3}{3}\right)\Big|_0^2 + (4x)\Big|_2^4 + \left(12x - x^2\right)\Big|_4^8 \\
&= \left(\frac{8}{3} - 0\right) + (16-8) + ((96-64) - (48-16)) \\
&= \frac{32}{3}
\end{aligned}
$$

Grading scheme: 1 point for breaking it into parts; 2/3 for solving each integral either by finding the area or by finding an anti-derivative

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Solution:

1. [2 points] Solve for x: $e^{2x-4} = 3^x$. (Hint: apply ln.)

Solution:

$$\begin{aligned} e^{2x-4} = 3^x &\Rightarrow \ln(e^{2x-4}) = \ln(3^x) \Rightarrow 2x - 4 = x\ln(3) \\ &\Rightarrow 2x - x\ln(3) = 4 \Rightarrow x(2 - \ln(3)) = 4 \\ &\Rightarrow x = \frac{4}{2 - \ln(3)} \end{aligned}$$

Grading scheme: Deduct 1 point for each error (to a minimum of 0).

2. [3 points] Evaluate the following derivatives:

(a) $\dfrac{d}{dx}\left(2x^5 + 7\sqrt{x} - 2e^x + \dfrac{3}{x^2}\right) =$

Solution:
We use the linearity of the derivative, that is, that $\frac{\mathrm{d}}{\mathrm{d}x}(f(x)+g(x)) = \frac{\mathrm{d}f}{\mathrm{d}x}(x) + \frac{\mathrm{d}g}{\mathrm{d}x}(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}(cf(x)) = c\frac{\mathrm{d}f}{\mathrm{d}x}(x)$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(2x^5 + 7\sqrt{x} - 2e^x + \frac{3}{x^2}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(2x^5\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(7\sqrt{x}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(2e^x\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3}{x^2}\right) \\ &= 2\frac{\mathrm{d}}{\mathrm{d}x}\left(x^5\right) + 7\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1/2}\right) - 2\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) + 3\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-2}\right) \\ &= 2\left(5x^4\right) + 7\left(\frac{1}{2}x^{-1/2}\right) - 2e^x + 3\left(-2x^{-3}\right) \\ &= 10x^4 + \frac{7}{2}x^{-1/2} - 2e^x - 6x^{-3} \end{aligned}$$

Grading scheme: Each derivative is worth 0.5; but deduct 1 point for "+c" (minimum 0)

(b) $\frac{d}{dx}\int_5^x \left(\frac{1}{\sqrt{t^3}} + 2e^t\right)\, dt =$

Solution:
We apply Part I of the Fundamental Theorem of Calculus, that is,
$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$ whenever f is a continuous function. Thus

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_5^x \left(\frac{1}{\sqrt{t^3}} + 2e^t\right)\,\mathrm{d}t = \frac{1}{\sqrt{x^3}} + 2e^x$$

Grading scheme: 1 point for correct answer (independent of method use to obtain it); deduct 0.5 for "+c"; deduct 0.5 for answer in t rather than x (minimum 0)

3. [2 points] Evaluate $\int_1^8 \left(8\sqrt[3]{x} - 5e^x + \frac{4}{x^3}\right) dx.$

Solution:
We use the linearity of the integral to simplify the question, that is, we use $\int_a^b (f(x) + g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$ and $\int_a^b (cf(x))\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$. Then we apply Part II of the Fundamental Theorem to evaluate the resulting simpler integrals.

$$\begin{aligned}
\int_1^8 \left(8\sqrt[3]{x} - 5e^x + \frac{4}{x^3}\right)\,\mathrm{d}x &= \int_1^8 \left(8\sqrt[3]{x}\right)\,\mathrm{d}x - \int_1^8 (5e^x)\,\mathrm{d}x + \int_1^8 \left(\frac{4}{x^3}\right)\,\mathrm{d}x \\
&= 8\int_1^8 \left(x^{1/3}\right)\,\mathrm{d}x - 5\int_1^8 (e^x)\,\mathrm{d}x + 4\int_1^8 \left(x^{-3}\right)\,\mathrm{d}x \\
&= 8\left(\frac{3}{4}x^{4/3}\right)\Big|_1^8 - 5\left(e^x\right)\Big|_1^8 + 4\left(-\frac{1}{2}x^{-2}\right)\Big|_1^8 \\
&= 6\left(8^{4/3} - 1\right) - 5\left(e^8 - e\right) - 2\left(8^{-2} - 1\right) = \frac{2943}{32} - 5\left(e^8 - e\right) \quad (\sim -14799.23)
\end{aligned}$$

Grading scheme: 0.5 for each of the 3 correct anti-derivatives and 0.5 for correct final

answer. Either decimal approximation or exact answer accepted. Write “simplify” if student does not simplify; points will be deducted on Midterm #2.

4. [2 points] Which of the following expressions is the **right Riemann sum with** n **subintervals** for estimating the value of the integral $\int_0^1 \sin(\sqrt{x})\, dx$?

A: $\sum_{i=1}^{n} \sin\left(\sqrt{\frac{i-1}{n}}\right)\frac{1}{n}$ **B**: $\sum_{i=0}^{n-1} \sin\left(\sqrt{\frac{i}{n}}\right)\frac{1}{n}$ **C**: $\sum_{i=1}^{n} \sin\left(\sqrt{\frac{i}{n}}\right)\frac{1}{n}$

D: $\sum_{i=1}^{n} \sin\left(\sqrt{\frac{i+1}{n}}\right)\frac{1}{n}$ **E**: $\sum_{i=1}^{n} \sin\left(\sqrt{\frac{i}{n}}\right)\frac{1}{2n}$ **F**: $\sum_{i=0}^{n-1} \sin\left(\sqrt{\frac{i}{n}}\right)\frac{1}{2n}$

Solution:
Since

$$f(x) = \sin(\sqrt{x})$$
$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

and

$$x_i = \frac{i}{n} \quad \text{pour} \quad i = 0, 1, 2, \ldots, n \ ,$$

we have

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \sin\left(\sqrt{\frac{i}{n}}\right)\frac{1}{n}$$

The correct answer is thus C.

Grading scheme: 2 points; no partial credit. Only one answer may be circled.

5. [2 points] Consider the velocity data of a car in the table below.

t (s)	0	3	6	9	12
v (m/s)	2	5	1	-2	-3

Compute the **left** Riemann sum with 4 subintervals of $\int_0^{12} v(t)\ dt$. Include the unit of measurement in your answer.

Solution:
Since $\Delta x = 3$, we have that

$$L_4 = (v(0) + v(3) + v(6) + v(9))\Delta x = 3(2 + 5 + 1 - 2) = 18 \text{ m}.$$

Grading scheme: 1 for correct formula/expression, 0.5 for correct number, 0.5 for correct units.

6. [2 points] Suppose

$$\int_0^5 f(x)\ dx = -5 \qquad \text{and} \qquad \int_2^5 f(x)\ dx = 3.$$

Evaluate $\displaystyle\int_0^2 (2f(x) - 4)\ dx$.

Solution:
We use the property that $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$. Here,

$$\int_0^2 f(x)\,\mathrm{d}x = \int_0^5 f(x)\,\mathrm{d}x - \int_2^5 f(x)\,\mathrm{d}x = -5 - 3 = -8$$

Thus, using the linearity of the integral, we have

$$\int_0^2 (2f(x) - 4)\,\mathrm{d}x = 2\int_0^2 f(x)\,\mathrm{d}x - \int_0^2 4\,\mathrm{d}x = 2(-8) - (4x)\Big|_0^2 = -16 - 8 = -24.$$

Grading scheme: 0.5 for formula to find the required integral of f; 0.5 for correct value; 0.5 for expanding the integral as a sum; 0.5 for final answer.

7. [4 points]

(a) State the definition of the derivative of a function f at an arbitrary point x in the domain.

Solution:
The derivative of f at a point x in the domain of f is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists.

Grading scheme: 1 point. Write "if limit exists" if this is missing but do not deduct.

(b) Use the definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{3}{2x+1}$. Write your solution correctly, logically, and clearly.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left(\dfrac{3}{2(x+h)+1} - \dfrac{3}{2x+1} \right)}{h} \\ &= \lim_{h \to 0} \frac{3(2x+1) - 3(2(x+h)+1)}{h(2(x+h)+1)(2x+1)} = \lim_{h \to 0} \frac{-6h}{h(2(x+h)+1)(2x+1)} \\ &= \lim_{h \to 0} \frac{-6}{(2(x+h)+1)(2x+1)} = \frac{-6}{(2x+1)^2} \end{aligned}$$

whenever $x \neq -1/2$, since $2(x+h)+1 \to 2x+1$ when $h \to 0$.

Grading scheme: 1 for correctly plugging the function into the formula for the difference quotient, 1 point for successful simplification of the difference quotient, 1 for correct final limit.

8. [3 points] We are given the function

$$g(x) = \begin{cases} -x^2 & \text{if } 0 \leq x < 2 \\ -4 & \text{if } 2 \leq x < 3 \\ 2x - 10 & \text{if } 3 \leq x \leq 6. \end{cases}$$

Evaluate $\int_0^6 g(x)\, dx$.

Solution:

$$
\begin{aligned}
\int_0^6 g(x)\,\mathrm{d}x &= \int_0^2 g(x)\,\mathrm{d}x + \int_2^3 g(x)\,\mathrm{d}x + \int_3^6 g(x)\,\mathrm{d}x \\
&= -\int_0^2 x^2\,\mathrm{d}x - \int_2^3 4\,\mathrm{d}x + \int_3^6 (2x-10)\,\mathrm{d}x \\
&= -\left(\frac{x^3}{3}\right)\Big|_0^2 - (4x)\Big|_2^3 + \left(x^2-10x\right)\Big|_3^6 \\
&= -\left(\frac{8}{3}-0\right) - (12-8) + ((36-60)-(9-30)) \\
&= -\frac{29}{3}
\end{aligned}
$$

Grading scheme: 1 point for breaking it into parts; 2/3 for solving each integral either by finding the area or by finding an anti-derivative

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Solution:

1. [2 points] Solve for x: $e^{4-x} = 2^x$. (Hint: apply ln.)

Solution:

$$\begin{aligned} e^{4-x} = 2^x &\Rightarrow \ln(e^{4-x}) = \ln(2^x) \Rightarrow 4 - x = x\ln(2) \\ &\Rightarrow 4 = x + x\ln(2) \Rightarrow 4 = x(1+\ln(2)) \\ &\Rightarrow x = \frac{4}{1+\ln(2)} \end{aligned}$$

Grading scheme: Deduct 1 point for each error (to a minimum of 0).

2. [3 points] Evaluate the following derivatives:

(a) $\dfrac{d}{dx}\left(5x^8 + 5\sqrt[3]{x} - 5e^x + \dfrac{3}{x}\right) =$

Solution:
We use the linearity of the derivative, that is, that $\frac{\mathrm{d}}{\mathrm{d}x}(f(x)+g(x)) = \frac{\mathrm{d}f}{\mathrm{d}x}(x) + \frac{\mathrm{d}g}{\mathrm{d}x}(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}(cf(x)) = c\frac{\mathrm{d}f}{\mathrm{d}x}(x)$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(5x^8 + 5\sqrt[3]{x} - 5e^x + \frac{3}{x}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(5x^8\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(5\sqrt[3]{x}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(5e^x\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3}{x}\right) \\ &= 5\frac{\mathrm{d}}{\mathrm{d}x}\left(x^8\right) + 5\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1/3}\right) - 5\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) + 3\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-1}\right) \\ &= 5\left(8x^7\right) + 5\left(\frac{1}{3}x^{-2/3}\right) - 5e^x + 3\left(-x^{-2}\right) \\ &= 40x^7 + \frac{5}{3}x^{-2/3} - 5e^x - 3x^{-2} \end{aligned}$$

Grading scheme: Each derivative is worth 0.5; but deduct 1 point for "+c" (minimum 0)

(b) $\dfrac{d}{dx}\displaystyle\int_1^x \left(3e^t - \dfrac{3}{\sqrt{t}}\right)\, dt =$

Solution:

We apply Part I of the Fundamental Theorem of Calculus, that is, that $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_a^x f(t)\,\mathrm{d}t = f(x)$ whenever f is a continuous function. Thus

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x \left(3e^t - \frac{3}{\sqrt{t}}\right)\,\mathrm{d}t = 3e^x - \frac{3}{\sqrt{x}}$$

Grading scheme: 1 point for correct answer (independent of method use to obtain it); deduct 0.5 for "+c"; deduct 0.5 for t instead of x (minimum 0)

3. [2 points] Evaluate $\displaystyle\int_1^4 \left(6\sqrt{x} - 2e^x + \frac{3}{x^2}\right) dx.$

Solution:

We use the linearity of the integral, that is, the properties that $\displaystyle\int_a^b (f(x) + g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$ and $\displaystyle\int_a^b (cf(x))\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$, to simplify the question, and then we apply Part II of the Fundamental Theorem to evaluate the resulting, simpler integrals.

$$\begin{aligned}
\int_1^4 \left(6\sqrt{x} - 2e^x + \frac{3}{x^2}\right)\,\mathrm{d}x &= \int_1^4 \left(6\sqrt{x}\right)\,\mathrm{d}x - \int_1^4 \left(2e^x\right)\,\mathrm{d}x + \int_1^4 \left(\frac{3}{x^2}\right)\,\mathrm{d}x \\
&= 6\int_1^4 \left(x^{1/2}\right)\,\mathrm{d}x - 2\int_1^4 \left(e^x\right)\,\mathrm{d}x + 3\int_1^4 \left(x^{-2}\right)\,\mathrm{d}x \\
&= 6\left(\frac{2}{3}x^{3/2}\right)\Bigg|_1^4 - 2\left(e^x\right)\Bigg|_1^4 + 3\left(-x^{-1}\right)\Bigg|_1^4 \\
&= 4\left(4^{3/2} - 1\right) - 2\left(e^4 - e\right) - 3\left(4^{-1} - 1\right) = \frac{121}{4} - 2\left(e^4 - e\right) \quad (\sim -73.51)
\end{aligned}$$

Grading scheme: 0.5 for each of the 3 correct anti-derivatives and 0.5 for correct final

answer. Either decimal approximation or exact answer accepted. Write "simplify" if student does not simplify; points will be deducted on Midterm #2.

4. [2 points] Which of the following expressions is the **left Riemann sum with** n **subintervals** for estimating the value of the integral $\int_0^1 \sin(\sqrt{x})\,dx$?

A: $\sum_{i=1}^{n} \sin\left(\sqrt{\frac{i+1}{n}}\right)\frac{1}{n}$ **B**: $\sum_{i=0}^{n-1} \sin\left(\sqrt{\frac{i}{n}}\right)\frac{1}{2n}$ **C**: $\sum_{i=1}^{n} \sin\left(\sqrt{\frac{i}{n}}\right)\frac{1}{n}$

D: $\sum_{i=1}^{n} \sin\left(\sqrt{\frac{i-1}{n}}\right)\frac{1}{n}$ **E**: $\sum_{i=1}^{n} \sin\left(\sqrt{\frac{i+1}{n}}\right)\frac{1}{2n}$ **F**: $\sum_{i=0}^{n-1} \sin\left(\sqrt{\frac{i-1}{n}}\right)\frac{1}{n}$

Solution:
Since

$$f(x) = \sin(\sqrt{x})$$
$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

and

$$x_i = \frac{i}{n} \quad \text{pour} \quad i = 0, 1, 2, \ldots, n \ ,$$

we have

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} \sin\left(\sqrt{\frac{i-1}{n}}\right)\frac{1}{n}$$

The correct answer is thus D.

Grading scheme: 2 points; no partial credit. Only one answer may be circled.

5. [2 points] Consider the velocity data of a car in the table below.

t (s)	0	3	6	9	12
v (m/s)	-7	-2	1	4	5

Compute the **right** Riemann sum with 4 subintervals of $\int_0^{12} v(t)\ dt$. Include the unit of measurement in your answer.

Solution:
Since $\Delta x = 3$, we have

$$R_4 = (v(3) + v(6) + v(9) + v(12))\,\Delta x = 3\,(-2 + 1 + 4 + 5) = 24\text{ m}$$

Grading scheme: 1 for correct formula/expression, 0.5 for correct number, 0.5 for correct units.

6. [2 points] Suppose

$$\int_0^7 f(x)\ dx = -6 \qquad \text{and} \qquad \int_4^7 f(x)\ dx = 3.$$

Evaluate $\displaystyle\int_0^4 (3f(x) - 2)\ dx$.

Solution:
We apply the following property of the integral: $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$. This gives

$$\int_0^4 f(x)\,\mathrm{d}x = \int_0^7 f(x)\,\mathrm{d}x - \int_4^7 f(x)\,\mathrm{d}x = -6 - 3 = -9$$

Thus, using the linearity of the integral, we deduce

$$\int_0^4 (3f(x) - 2)\,\mathrm{d}x = 3\int_0^4 f(x)\,\mathrm{d}x - \int_0^4 2\,\mathrm{d}x = 3(-9) - (2x)\big|_0^4 = -27 - 8 = -35.$$

Grading scheme: 0.5 for formula to find the required integral of f; 0.5 for correct value; 0.5 for expanding the integral as a sum; 0.5 for final answer.

7. [4 points]

(a) State the definition of the derivative of a function f at an arbitrary point x in the domain.

Solution:
The derivative of f at a point x in the domain of f is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists.

Grading scheme: 1 point. Write "if limit exists" if this is missing but do not deduct.

(b) Use the definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{4}{5x-1}$. Write your solution correctly, logically, and clearly.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left(\dfrac{4}{5(x+h)-1} - \dfrac{4}{5x-1}\right)}{h} \\ &= \lim_{h\to 0} \frac{4(5x-1)-4(5(x+h)-1)}{h(5(x+h)-1)(5x-1)} = \lim_{h\to 0} \frac{-20h}{h(5(x+h)-1)(5x-1)} \\ &= \lim_{h\to 0} \frac{-20}{(5(x+h)-1)(5x-1)} = \frac{-20}{(5x-1)^2} \end{aligned}$$

whenever $x \neq 1/5$ since $5(x+h)-1 \to 5x-1$ when $h \to 0$.

Grading scheme: 1 for correctly plugging the function into the formula for the difference quotient, 1 point for successful simplification of the difference quotient, 1 for correct final limit.

8. [3 points] We are given the function

$$g(x) = \begin{cases} 2x^2 & \text{if } 0 \le x < 1 \\ 2 & \text{if } 1 \le x < 4 \\ -2x+10 & \text{if } 4 \le x \le 8. \end{cases}$$

Evaluate $\int_0^8 g(x)\,dx$.

Solution:

$$
\begin{aligned}
\int_0^8 g(x)\,\mathrm{d}x &= \int_0^1 g(x)\,\mathrm{d}x + \int_1^4 g(x)\,\mathrm{d}x + \int_4^8 g(x)\,\mathrm{d}x \\
&= \int_0^1 2x^2\,\mathrm{d}x + \int_1^4 2\,\mathrm{d}x + \int_4^8 (-2x+10)\,\mathrm{d}x \\
&= 2\left(\frac{x^3}{3}\right)\Bigg|_0^1 + (2x)\Bigg|_1^4 + \left(-x^2+10x\right)\Bigg|_4^8 \\
&= 2\left(\frac{1}{3}-0\right) + (8-2) + ((-64+80)-(-16+40)) \\
&= -\frac{4}{3}
\end{aligned}
$$

Grading scheme: 1 point for breaking it into parts; 2/3 for solving each integral either by finding the area or by finding an anti-derivative