SOLUTION

a) $y^{(4)} - 2y^{(3)} + 5y^{(2)} = 0$

RE $r^4 - 2r^3 + 5r^2 = 0$ ② - work

$r^2(r^2 - 2r + 5) = 0$

$\hookrightarrow r = \dfrac{2 \pm \sqrt{4 - 4(1)(5)}}{2} = \dfrac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$

$\hookrightarrow r = 0, 0$

$\therefore y_h = c_1 e^{0x} + c_2 x e^{0x} + e^{x}(c_3 \cos(2x) + c_4 \sin(2x))$

$= c_1 + c_2 x + c_3 e^{x}\cos(2x) + c_4 e^{x}\sin(2x)$ ② solution

b) $y^{(5)} + 2y^{(4)} - 16y' - 32y = 0$

RE $0 = r^5 + 2r^4 - 16r - 32 = r^4(r+2) - 16(r+2)$

$= (r+2)(r^4 - 16) = (r+2)(r^2 - 4)(r^2 + 4)$ ② work

$= \underbrace{(r+2)(r+2)}\underbrace{(r-2)}\underbrace{(r+2i)(r-2i)}$

$\therefore y_h = c_1 e^{-2x} + c_2 x e^{-2x} + c_3 e^{2x} + c_4 \cos(2x) + c_5 \sin(2x)$

② solution

SOLUTION

$$\vec{x}' = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \vec{x}$$

$$0 = \begin{vmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 6 = 12 - 7\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 7\lambda + 6 = (\lambda-1)(\lambda-6) \Rightarrow \lambda = 1, 6 \quad 2$$

$\lambda_1 = 1$ // $\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2/3 \\ 0 & 0 \end{pmatrix}$ $\begin{cases} x_1 + \frac{2}{3}x_2 = 0 \\ x_2 = t \end{cases} = \begin{cases} x_1 = -\frac{2}{3}t \\ x_2 = t \end{cases}$

$$\Rightarrow \vec{v}_{\lambda_1} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad 1$$

$\lambda_2 = 6$ // $\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ $\begin{cases} x_1 - x_2 = 0 \\ x_2 = t \end{cases} = \begin{cases} x_1 = t \\ x_2 = t \end{cases}$

$$\Rightarrow \vec{v}_{\lambda_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad 1$$

$$\therefore \vec{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = PD^{*}\vec{c} = \begin{pmatrix} -2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} e^{t} & 0 \\ 0 & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad 1 \quad 1$$

$$= \begin{pmatrix} -2e^{t} & e^{6t} \\ 3e^{t} & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2c_1e^{t} + c_2e^{6t} \\ 3c_1e^{t} + c_2e^{6t} \end{pmatrix}$$

So $\begin{cases} x(t) = -2c_1e^{t} + c_2e^{6t} & \leftarrow 1 \\ y(t) = 3c_1e^{t} + c_2e^{6t} & \leftarrow 1 \end{cases}$

SOLUTIONS

(a) $\left\{ \dfrac{3n^2-2n-3}{\sqrt{2n^4+3n+1}} \right\}_{n=1}^{\infty}$

① - work

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2-2n-3}{\sqrt{2n^4-3n+1}} = \lim_{n\to\infty} \frac{\frac{3n^2}{n^2} - \frac{2n}{n^2} - \frac{3}{n^2}}{\sqrt{\frac{2n^4}{n^4} - \frac{3n}{n^4} + \frac{1}{n^4}}}$$

$$= \lim_{n\to\infty} \frac{3 - \overset{\to 0}{2/n} - \overset{\to 0}{3/n^2}}{\sqrt{2 - \underset{\to 0}{3/n^3} + \underset{\to 0}{\frac{1}{n^4}}}} = \underline{\underline{\frac{3}{\sqrt{2}}}}$$

So the sequence converges to $\frac{3}{\sqrt{2}}$ ①

(b) $\left\{ \sqrt{n^3+1} - n \right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{n^3+1} - n = \lim_{n\to\infty} \frac{\sqrt{n^3+1} - n}{1} \cdot \frac{\sqrt{n^3+1} + n}{\sqrt{n^3+1} + n}$$ ①

$$= \lim_{n\to\infty} \frac{n^3+1-n^2}{\sqrt{n^3+1} + n} = \lim_{n\to\infty} \frac{\frac{n^3}{n^3} + \frac{1}{n^3} - \frac{n^2}{n^3}}{\sqrt{\frac{n^3}{n^6} + \frac{1}{n^6}} + \frac{n}{n^6}}$$ ① - work

$$= \lim_{n\to\infty} \frac{1 + \overset{\to 0}{\frac{1}{n^3}} - \overset{\to 0}{\frac{1}{n^2}}}{\sqrt{\underset{\to 0}{\frac{1}{n^3}} + \underset{\to 0}{\frac{1}{n^6}}} + \underset{\to 0}{\frac{1}{n^5}}} = \underline{\underline{+\infty}}$$

So the sequence diverges. ①

SOLUTIONS

(c)

$$\left\{ \frac{e^n}{n!} \right\}_{n=1}^{\infty} \qquad \lim_{n\to\infty} \frac{e^n}{n!} = ?$$

$a_1 = \frac{e}{1}$

$a_2 = \frac{e^2}{2!} = \frac{e}{1} \cdot \frac{e}{2}$

$a_3 = \frac{e^3}{3!} = \frac{e}{1} \cdot \frac{e}{2} \cdot \frac{e}{3}$

$\vdots$

$a_n = \frac{e^n}{n!} = \frac{e}{1} \cdot \frac{e}{2} \cdot \frac{e}{3} \cdot \ldots \cdot \frac{e}{n} = \frac{e}{1} \cdot \frac{e}{2} \cdot$ "a bunch ~~of~~ fractions" $\cdot \frac{e}{n}$

$\leq \frac{e}{1} \cdot \frac{e}{2} \cdot \frac{e}{n} = \frac{e^3}{2n}$ 1

So, $\{a_n\}$ is bound ABOVE by $\left\{\frac{e^3}{2n}\right\}$, BELOW by $\{0\}$

Then $0 \leq a_n \leq \frac{e^3}{2n}$

$$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} \frac{e^3}{2n} = 0$$ 1

$\therefore$ Since $0 \leq \lim_{n\to\infty} a_n \leq 0$, by the Sandwich Theorem, $\lim_{n\to\infty} \frac{e^n}{n!} = 0$ 1

The sequence converges to 0.