Lesson 17 Assessment Answers

Task 1: Knowledge and Understanding questions

1. Determine the average rate of change for the following points. **(6 marks: 2 marks each)**

   a) $A(8,32)$ and $B(12,96)$

   Answer:

   Average rate of change = $\frac{96-32}{12-8}=\frac{64}{4}=16$

   b) $P(20,88)$ and $Q(22,48)$

   Answer:

   Average rate of change = $\frac{48-88}{22-20}=\frac{-40}{2}=-20$

   c) $S(-3,7)$ and $T(5,-17)$

   Answer:

   Average rate of change = $\frac{-17-7}{5+3}=\frac{-24}{8}=-3$

2. Determine the instantaneous rate of change at $P(2,-1)$ for the graph of $y=\frac{2}{x-4}$. Round all non-terminating decimals in the calculation to seven decimal places. Round your final answer to two decimal places. **(5 marks)**

   Answer:

   First point:

   $x = 2 + 0.01 = 2.01$

   $y=\frac{2}{(2.01-4)}=-1.0050251$

   The first point is $(2.01, -1.0050251)$

   Second point:

   $x = 2 - 0.01 = 1.99$

   $y=\frac{2}{(1.99-4)}= -0.9950249$

   The second point is $(1.99, -0.9950249)$

   Slope $=\frac{-1.0050251-(-0.9950249)}{2.01-1.99}=\frac{-0.0100002}{0.02}=-0.5000113$

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The instantaneous rate of change is –0.50.

3. The slope of a tangent line is 6. The point of tangency is $(-3,-14)$. Determine the equation of the tangent line. **(2 marks)**

Answer:

$y = m(x - x_1) + y_1$

$y = 6(x + 3) - 14$

$y = 6x + 18 - 14$

$y = 6x + 4$

4. By calculating the instantaneous rate of change, prove that $(4,-7)$ is the vertex of the function, $y = 2x^2 - 16x + 25$. **(5 marks)**

Answer:

First point:

$x = 4 + 0.01 = 4.01$

$y = 2(4.01)^2 - 16(4.01) + 25 = -6.9998$

The first point is $(4.01,-6.9998)$.

Second point:

$x = 4 - 0.01 = 3.99$

$y = 2(3.99)^2 - 16(3.99) + 25 = -6.9998$

The second point is $(3.99,-6.9998)$.

Slope $= \dfrac{-6.9998 - (-6.9998)}{4.01 - 3.99} = \dfrac{0}{0.02} = 0$

The instantaneous rate of change is 0. Therefore, $(4,-7)$ is the vertex of the function, $y = 2x^2 - 16x + 25$.

5. Determine the instantaneous rate of change at $x = 4$ for the graph of $y = 7\cos x$. Round all non-terminating decimals in the calculation to seven decimal places. Round your final answer to two decimal places. **(5 marks)**

Answer:

First point:

$x = 4 + 0.01 = 4.01$

$y = 7\cos(4.01) = -4.522301$

The first point is $(4.01, -4.522301)$.

Second point:

$x = 4 - 0.01 = 3.99$

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$y = 7\cos(3.99) = -4.628252$

The second point is $(3.99, -4.628252)$.

Slope $= \dfrac{-4.522301 - (-4.628252)}{4.01 - 3.99} = \dfrac{0.105951}{0.02} = 5.29755$

The instantaneous rate of change is 5.30.

Task 2: Thinking questions

6. Determine whether $y = 10x - 12$ is a tangent or a secant line to the graph of $y = 2x^2$. Write the equation of the line. **(6 marks)**

Answer:

Use substitution to determine how many points of intersection there are for the two graphs.

$2x^2 = 10x - 12$

$2x^2 - 10x + 12 = 0$

$2(x^2 - 5x + 6) = 0$

$2(x - 2)(x - 3) = 0$

$x = 2 \text{ or } x = 3$

When $x = 2$:

$y = 10x - 12$

$y = 10(2) - 12 = 20 - 12 = 8$

The first point of intersection is $(2,8)$.

When $x = 3$:

$y = 10(3) - 12 = 30 - 12 = 18$

The second point of intersection is (3,18).

There are two points of intersection. Therefore, this is a secant line.
To write the equation of the secant line, we need the slope, and a point on the line.

Slope $= \dfrac{18 - 8}{3 - 2} = \dfrac{10}{1} = 10$

There are two endpoints. Choose either one.

$y = m(x - x_1) + y_1$

$y = 10(x - 8) + 2$

$y = 10x - 80 + 2$

$y = 10x - 78$

The equation of the secant line is $y = 10x - 78$.

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7. Determine the point of tangency on the graph of $y = 3x^2 - 18x + 11$, if the equation of the tangent line is $y = 12x - 64$. **(4 marks)**

Answer:

Use substitution to find the point of tangency.

$3x^2 - 18x + 11 = 12x - 64$

$3x^2 - 18x + 11 - 12x + 64 = 0$

$3x^2 - 30x + 75 = 0$

$3(x^2 - 10x + 25) = 0$

$3(x - 5)(x - 5) = 0$

$x = 5$

$y = 12(5) - 64 = 60 - 64 = -4$

The point of tangency is $(5, -4)$.

8. A rabbit population doubles every month. The initial number of rabbits was 14.

a) What is the average rate of change in the number of rabbits during the twelfth month? **(4 marks)**

b) What is the instantaneous rate of change in the number of rabbits at exactly 1 year? Round your answer to the nearest whole number. **(4 marks)**

Answer:

The equation for doubling time in months, with an initial amount of 14, is $y = 14(2)^t$.

a) At the end of 11 months:

$y = 14(2)^{11} = 28\ 672$

This is represented by the point $(11, 28\ 672)$.

At the end of 12 months:

$y = 14(2)^{12} = 57\ 344$

This is represented by the point $(12, 57\ 344)$.

Slope $= \dfrac{57\ 344 - 28\ 672}{12 - 11} = \dfrac{28\ 672}{1} = 28\ 672$

The average rate of change in the twelfth month is 28 672 rabbits per month.

b) 1 year = 12 months

First point:

$12 + 0.01 = 12.01$

$y = 14(2)^{12.01} = 57\ 743$

The first point is $(12.01, 57\ 743)$.

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Second point:

$12 - 0.01 = 11.99$

$y = 14(2)^{11.99} = 56\ 948$

The second point is $(11.99, 56\ 948)$.

Slope = $\dfrac{57\ 743 - 56\ 948}{12.01 - 11.99} = \dfrac{795}{0.02} = 39\ 750$

The instantaneous rate of change at exactly 1 year is $39\ 750$ rabbits per year.

9. The line tangent to $y = -x^3 + 2x + 1$ at $x = 1$ intersects the graph at another point. Find the coordinates of the other point. **(8 marks)**

Answer:

Find the slope of the tangent line. **(3 marks**)

First point:

$x = 1 + 0.01 = 1.01$

$y = -(1.01)^3 + 2(1.01) + 1 = 1.989699$

The first point is $(1.01, 1.989699)$.

Second point:

$x = 1 - 0.01 = 0.99$

$y = -(0.99)^3 + 2(0.99) + 1 = 2.009701$

The second point is $(0.99, 2.009701)$.

Slope = $\dfrac{1.989699 - 2.009701}{1.01 - 0.99} = \dfrac{-0.020002}{0.02} = -1.0001$

The slope of the tangent line is –1.

Find the y-value of the point of tangency. **(1 mark)**

$y = -(1)^3 + 2(1) + 1 = -1 + 2 + 1 = 2$

The point of tangency is $(1,2)$.

Determine the equation of the tangent line. **(1 mark)**

$$y = -1(x-1) + 2$$
$$y = -1x + 1 + 2$$
$$y = -1x + 3$$

Determine the points of intersection of the tangent line and the graph. **(2 marks)**

$$-1x + 3 = -x^3 + 2x + 1$$
$$x^3 - 1x - 2x + 3 - 1 = 0$$
$$x^3 - 3x + 2 = 0$$

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Factor using the remainder theorem and the factor theorem:

$(x-1)(x^2+x-2)=0$
$(x-1)(x-1)(x+2)=0$
$x=1 \text{ or } x=-2$

The point of tangency is at $x=1$. Therefore, the other point has an x-value of -2.

Calculate the y-value for the intersection point. **(1 mark)**

$y=-1x+3$
$y=-1(-2)+3=2+3=5$

The other intersection point is $(-2,5)$.

Task 3: Communication questions

10. Describe the difference between a secant line and a tangent line. **(4 marks: 2 marks each)**

Answer:

A secant line is a straight line that intersects a graph at two or more points. The slope of a secant line between two intersection points represents the average rate of change between these points.

A tangent line is a straight line that touches a curve at one point, called a tangent point. A tangent line can also intersect a graph at other points, but there is only one point of tangency. The slope of a tangent line represents the instantaneous rate of change.

11. How can tangent lines be used in curve sketching? **(5 marks)**

Answer:

- The slopes of the tangent lines along a curve can indicate where a graph is increasing or decreasing.
- When the slopes of the tangent lines are negative, the graph is decreasing.
- When the slopes of the tangent lines are positive, the graph is increasing.
- The slope of a tangent line can confirm the location of any maximum or minimum points.
- At the local maximum and minimum points, the slopes of the tangent lines are 0.

12. The function $h(t)=-3t^2+12t+1$ models the path of a baseball, where $h(t)$ represents the height of the ball above the ground in metres, and t represents the time in seconds.

a) Calculate the instantaneous rate of change at $t=2$ and use it to prove that $(2,13)$ is the vertex of the graph. **(3 marks)**

b) For what interval is the baseball moving upwards? For what interval is the baseball moving downwards? **(6 marks)**

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c) What is the instantaneous velocity of the ball when it hits the ground? **(6 marks)**

Answer:

a) First point:

$t = 2 + 0.01 = 2.01$

$h(2.01) = -3(2.01)^2 + 12(2.01) + 1 = 12.9997$

The first point is $(2.01, 12.9997)$.

Second point:

$t = 2 - 0.01 = 1.99$

$h(1.99) = -3(1.99)^2 + 12(1.99) + 1 = 12.9997$

The first point is $(1.99, 12.9997)$.

Slope = $\dfrac{12.9997 - 12.9997}{2.01 - 1.99} = \dfrac{0}{0.02} = 0$

The point $(2,13)$ is the vertex because the instantaneous rate of change at $t = 2$ is 0 m/s.

b) Test a point on the left of $(2,13)$:

At $t = 1$:

First point:

$t = 1 + 0.01 = 1.01$

$h(1.01) = -3(1.01)^2 + 12(1.01) + 1 = 10.0597$

The first point is $(1.01, 10.0597)$.

Second point:

$t = 1 - 0.01 = 0.99$

$h(0.99) = -3(0.99)^2 + 12(0.99) + 1 = 9.9397$

The second point is (0.99, 9.9397).

Slope = $\dfrac{10.0597 - 9.9397}{0.01 - (-0.01)} = \dfrac{0.12}{0.02} = 6$

The instantaneous rate of change is positive. Therefore, the baseball is moving upwards until it reaches the vertex. Since the vertex occurs at $t = 2$, the interval where the graph is increasing is $0 < t < 2$.

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Test a point on the right of $(2,13)$:

At $t = 3$:

First point:

$t = 3 + 0.01 = 3.01$

$h(3.01) = -3(3.01)^2 + 12(3.01) + 1 = 9.9397$

The first point is $(3.01, 9.9397)$.

Second point:

$t = 3 - 0.01 = 2.99$

$h(2.99) = -3(2.99)^2 + 12(2.99) + 1 = 10.0597$

The second point is $(2.99, 10.0597)$.

Slope = $\dfrac{10.0597 - 9.9397}{0.01 - (-0.01)} = \dfrac{0.12}{0.02} = 6$

The instantaneous rate of change is negative. Therefore, the baseball is moving downwards after the vertex. The interval where the graph is decreasing is $t > 2$.

c) Calculate the time at which the ball hits the ground. This occurs when $h(t)$, the height of the ball above the ground, is 0. Round your answer to two decimal places.

At $h(t) = 0$:

$0 = -3t^2 + 12t + 1$

Use the quadratic formula to solve for the variable t, where $a = -3$, $b = 12$, and $c = 1$:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-12 \pm \sqrt{(12)^2 - 4(-3)(1)}}{2(-3)} = \frac{-12 \pm \sqrt{156}}{-6}$$

$t = -0.08$ or $t = 4.08$

The ball hits the ground at 4.08 seconds.

Determine the instantaneous rate of change at 4.08 seconds.

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First point:

$t = 4.08 + 0.01 = 4.09$

$h(4.09) = -3(4.09)^2 + 12(4.09) + 1 = -0.1043$

The first point is $(4.09, -0.1043)$.

Second point:

$t = 4.08 - 0.01 = 4.07$

$h(4.07) = -3(4.07)^2 + 12(4.07) + 1 = 0.1453$

The second point is $(4.07, 0.1453)$.

Slope $= \dfrac{-0.1043 - 0.1453}{4.09 - 4.07} = \dfrac{-0.2496}{0.02} = -12.48$

The instantaneous velocity is 12.48 m/s downwards.

Task 4: Application questions

13. An amount of $1500 is invested in an RESP at 5% per year, compounded annually. The following table shows the value of the investment at the end of each year, for a 5-year period. **(15 marks)**

Year	Amount of investment ($)
1	1575
2	1654
3	1736
4	1823
5	1914

a) What is the average rate of change, rounded to the nearest dollar, for the:
   i) first year?
   ii) second year?
   iii) third year?
   iv) fourth year?
   v) fifth year?

b) Why are the rates different?

c) Sketch a graph to illustrate the data.

d) What type of function does the graph represent?

e) Write an equation to model the growth of the investment.

f) Calculate the average rate of growth during the eighth year. Round your answer to the nearest dollar.

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Answer:

a)

i) Rate of change from $(0{,}1500)$ to $(1{,}1575)$:

Rate of growth $= \dfrac{1575-1500}{1-0} = \dfrac{75}{1} = 75$

During the first year, the investment grows at a rate of $75 per year.

ii) Rate of change from $(1{,}1575)$ to $(2{,}1654)$:

Rate of growth $= \dfrac{1654-1575}{1-0} = \dfrac{79}{1} = 79$

During the second year, the investment grows at a rate of $79 per year.

iii) Rate of change from $(2{,}1654)$ to $(3{,}1736)$:

Rate of growth $= \dfrac{1736-1654}{1-0} = \dfrac{82}{1} = 82$

During the third year, the investment grows at a rate of $82 per year.

iv) Rate of change from $(3{,}1736)$ to $(4{,}1823)$:

Rate of growth $= \dfrac{1823-1736}{1-0} = \dfrac{87}{1} = 87$

During the fourth year, the investment grows at a rate of $87 per year.

v) Rate of change from $(4{,}1823)$ to $(5{,}1914)$:

Rate of growth $= \dfrac{1914-1823}{1-0} = \dfrac{91}{1} = 91$

During the fifth year, the investment grows at a rate of $91 per year.

b) The rates are different because the amount of interest earned is proportional to the initial amount of money invested at the beginning of each year.

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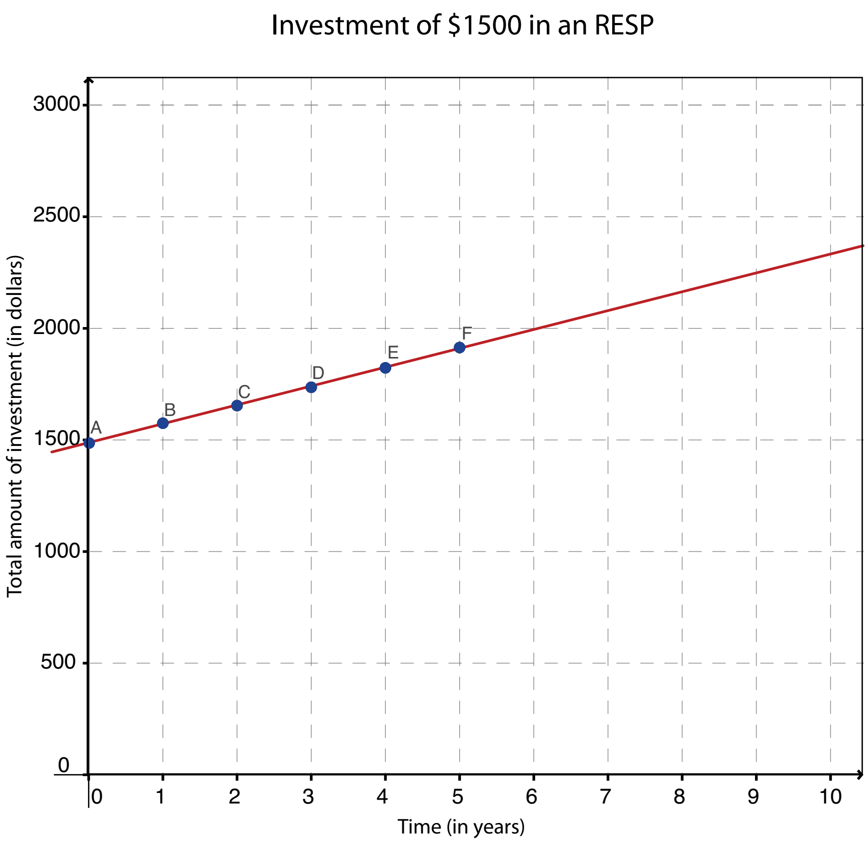
c) Graph:



d) The graph represents exponential growth.

e) The formula for compound interest can be used to model the growth of this investment.

The formula is $A = P(1 + i)^n$, where:

P = initial amount of the investment

A = the final amount of the investment

i = the interest rate per compounding period

n = the number of compounding periods

Since the initial amount of the investment is \$1500, $P = 1500$. The interest rate as a decimal is 0.05. Therefore, $i = 0.05$. The equation that represents the growth of the investment is $A = 1500(1.05)^n$.

f) Determine the total amount of the investment at the end of the seventh and eighth years.

At the end of the seventh year, the total amount of the investment is approximately:

$A = 1500(1.05)^7 = \$2111$

This is represented by the point $(7,2111)$.

At the end of the eighth year, the total amount of the investment is:

$A = 1500(1.05)^8 = \$2216$

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This is represented by the point $(8,2216)$.

The average rate of growth from $(7,2111)$ to $(8,2216)$ is:

Rate of growth $= \dfrac{2216 - 2111}{8 - 7} = \dfrac{105}{1} = 105$

The average rate of growth during the tenth year is $105 per year.

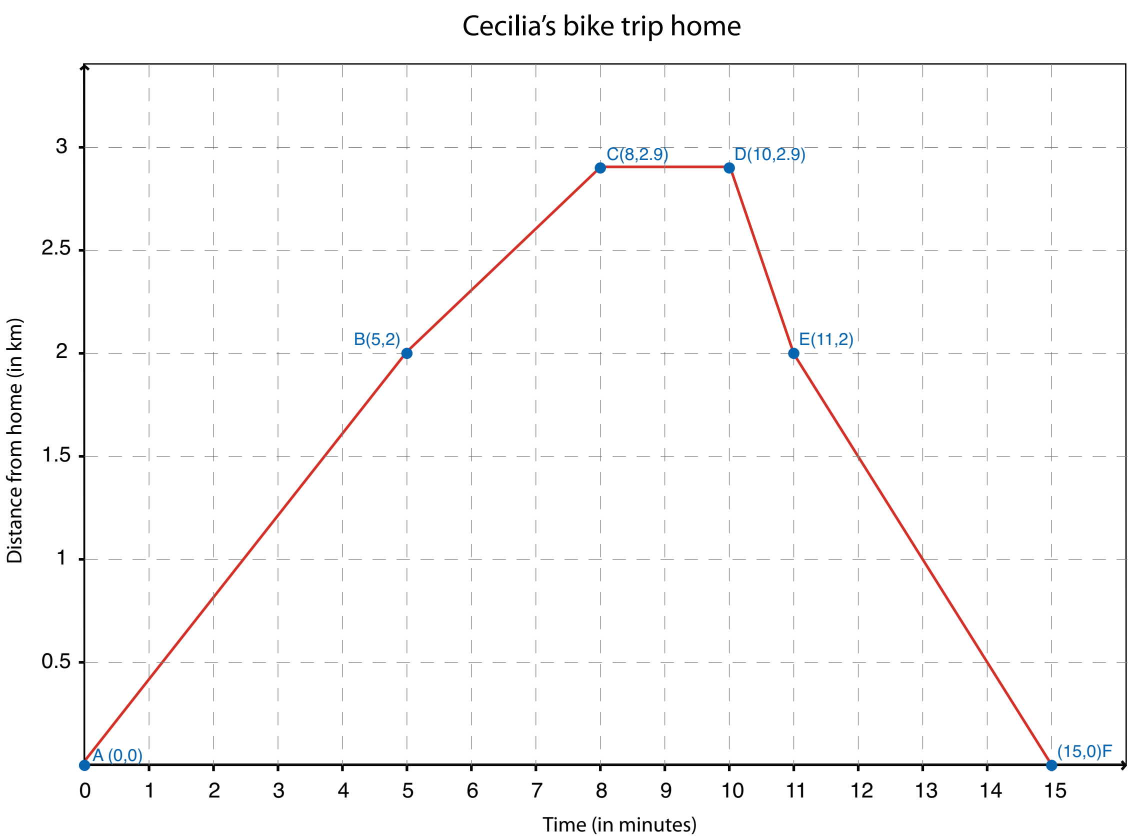
14. Cecilia is riding her bike as part of a fitness program. She rides on a flat road at a constant rate of 0.4 km/min for 5 minutes. She then rides up a hill at a rate of 0.3 km/min for 3 minutes. She stops at the top of the hill for 2 minutes. After this brief rest, she rides back down the same side of the hill towards home at a rate of 0.9 km/min for 1 minute, and finishes the trip home at a rate of 0.5 km/min. **(12 marks)**

a) Sketch a graph to show Cecilia's distance from home in relation to time, in minutes.

b) Determine all of the relevant equations and intervals.

Answer:

a) Graph:



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b) Line segment AB:

Slope = 0.4 km/min

Point: (0,0)

$y = 0.4(x - 0) + 0$

$y = 0.4x$

Interval: $0 \le x \le 5$

Line segment BC:

Slope = 0.3 km/min

Point: $(5,2)$

$y = 0.3(x - 5) + 2$

$y = 0.3x - 1.5 + 2$

$y = 0.3x + 0.5$

Interval: $5 \le x \le 8$

Line segment CD:

Slope = 0 km/min

Point: $(8,2.9)$

$y = 0(x - 8) + 2.9$

$y = 2.9$

Interval: $8 \le x \le 10$

Line segment DE:

Slope = –0.9 km/min

Point: $(11,2)$

$y = -0.9(x - 11) + 2$

$y = -0.9x + 9.9 + 2$

$y = -0.9x + 11.9$

Interval: $10 \le x \le 11$

Line segment EF:

Slope = –0.5 km/min

Point: $(15,0)$

$y = -0.5(x - 15) + 0$

$y = -0.5x + 7.5$

Interval: $11 \le x \le 15$

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